Homework 10

Section 11.10

$\boxed{16}$ $f(x) = x\cos x$

$f'(x) = \cos x - x\sin x$

$f''(x) = -\sin x - \sin x - x\cos x = -2\sin x - x\cos x$

$f^{(3)}(x) = -2\cos x - \cos x + x\sin x = -3\cos x + \cancel{x\cos x}\, x\sin x$

$f^{(4)}(x) = 3\sin x + \sin x + x\cos x = 4\sin x + x\cos x$

$\vdots$

given : $f^{(2n)}(x) = (-1)^n 2n\sin x + (-1)^n x\cos x$

$f^{(2n+1)}(x) = (-1)^n (2n+1)\cos x - (-1)^n x\sin x$

You can show by induction the above equalities are correct.

Thus $f^{(2n)}(0) = 0$

$f^{(2n+1)}(0) = (-1)^n (2n+1)$

$$\rightsquigarrow x\cos x = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$(2n+1)! = (2n)!\,(2n+1)$

$\boxed{20}$ $f(x) = x^6 - x^4 + 2$

Solution 1

$f'(x) = 6x^5 - 4x^3$

$f''(x) = 30x^4 - 12x^2$

$f'''(x) = 120x^3 - 24x$

$f^{(4)}(x) = 360x^2 - 24$

$f^{(5)}(x) = 720x$

$f^{(6)}(x) = 720$

$f(-2) = 64 - 16 + 2 = 50$

$f'(-2) = -192 + 32 = -160$

$f''(-2) = 480 - 48 = 432$

$f'''(-2) = -960 + 48 = -912$

$f^{(4)}(-2) = 1440 - 24 = 1416$

$f^{(5)}(-2) = -1440$

$f^{(6)}(-2) = 720$

$$\rightsquigarrow f(x) = 50 - 160(x+2) + \frac{432}{2}(x+2)^2 - \frac{912}{3!}(x+2)^3 + \frac{1416}{4!}(x+2)^4$$

$$- \frac{1440}{5!}(x+2)^5 + \frac{720}{6!}(x+2)^6$$

$$= 50 - 160(x+2) + 216(x+2)^2 - 152(x+2)^3 + 59(x+2)^4$$

$$- 12(x+2)^5 + (x+2)^6$$

It's a polynomial and has finitely many terms. So $R = \infty$ i.e. it's convergent for any real number x.

Solution 2

Let $t = x+2 \rightsquigarrow x = t-2$

$$(t-2)^6 - (t-2)^4 + 2 = \left(t^6 - 12t^5 + 60t^4 - 160t^3 + 240t^2 - 192t + 64\right)$$

$$- (t^4 - 8t^3 + 24t^2 - 32t + 16)$$

$$+ 2$$

find Taylor series at $a = -2$ by substitution from Maclaurin series

expand $(t-2)^6$ and $(t-2)^4$

$$= t^6 - 12t^5 + 59t^4 - 152t^3 + 216t^2 - 160t + 50 \quad (*)$$

$(*)$ is a polynomial, therefore its Maclaurin series is equal to itself i.e.

Maclaurin Series $= 50 - 160t + 216t^2 - 152t^3 + 59t^4 - 12t^5 + t^6 \qquad R = \infty$

$$\xrightarrow{\text{Taylor series } f(x) \text{ at } a=-2} = 50 - 160(x+2) + 216(x+2)^2 - 152(x+2)^3 + 59(x+2)^4 - 12(x+2)^5$$

$$+ (x+2)^6 \qquad R = \infty$$

$\boxed{24}$ <u>Solution 1</u>

$$f(x) = \cos x \qquad f\left(\frac{\pi}{2}\right) = 0$$

$$f'(x) = -\sin x \qquad f'\left(\frac{\pi}{2}\right) = -1$$

$$f''(x) = -\cos x \qquad f''\left(\frac{\pi}{2}\right) = 0$$

$$f^{(3)}(x) = \sin x \qquad f^{(3)}(x) = 1$$

$$\rightsquigarrow f(x) = 0 - \left(x - \frac{\pi}{2}\right) + 0 + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 + 0 - \frac{1}{5!}\left(x - \frac{\pi}{2}\right)^5 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x - \frac{\pi}{2}\right)^{2n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+3}}{(2n+3)!}\left(x-\frac{\pi}{2}\right)^{2n+3} \cdot \frac{(2n+1)!}{(-1)^{n+1}} \cdot \frac{1}{\left(x-\frac{\pi}{2}\right)^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\left(x-\frac{\pi}{2}\right)^2}{(2n+2)(2n+3)}\right| = 0 < 1 \quad \rightsquigarrow \quad R = \infty$$

<u>Solution 2</u> $\quad t = x - \frac{\pi}{2} \rightsquigarrow x = t + \pi/2$

$$\rightsquigarrow \cos x = \cos\left(t + \frac{\pi}{2}\right) = -\sin t = -\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$\rightsquigarrow \cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\left(x - \pi/2\right)^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$\boxed{32}$

$$f(x) = \sqrt[3]{8+x} = (8+x)^{1/3} = 2\left(1 + \frac{x}{8}\right)^{1/3}$$

$$2\left(1+\frac{x}{8}\right)^{1/3} = 2\sum_{n=0}^{\infty}\binom{1/3}{n}\left(\frac{x}{8}\right)^n = \sum_{n=0}^{\infty}\binom{1/3}{n}\frac{2}{\underbrace{8^n}_{2^{3n}}}x^n$$

Conv. for

$$\left|\frac{x}{8}\right| < 1 \qquad = \sum_{n=0}^{\infty}\binom{1/3}{n}\frac{x^n}{2^{3n-1}}$$

$$\rightsquigarrow |x| < 8 \rightsquigarrow \boxed{R = 8}$$

[36] $f(x) = \sin\left(\frac{\pi x}{4}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi x}{4}\right)^{2n+1}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty} \cdots \frac{\cdots^{2n+1}}{\cdots (2n+1)!}$$

[64] $\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2} =$

$\left[\text{Note that: } \sqrt{1+x} = (1+x)^{1/2} \underset{|x|<1}{=} \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots\right]$

$$\rightsquigarrow \lim_{x \to 0} \frac{\left[1 + \frac{x}{2} - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots\right] - 1 - \frac{1}{2}x}{x^2} = \lim_{x \to 0} \frac{-\frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots}{x^2}$$

$$= \lim_{x \to 0} -\frac{1}{8} + \frac{1}{16}x + \cdots = \boxed{-\frac{1}{8}}$$

[77] $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1} = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$\left(\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right)$$

[81] P is a nth degree polynomial, thus $P^{(i)} = 0$ for any $i \geq n+1$

Therefore Taylor series of P at any $a = x$ is a degree n polynomial.

$$P(y) \underset{\substack{\text{Taylor series} \\ \text{at } a = x}}{=} \sum_{i=0}^{n} \frac{P^{(i)}(x)}{i!} (y-x)^i \quad \underset{y = x+1}{\rightsquigarrow} \quad P(x+1) = \sum_{i=0}^{n} \frac{P^{(i)}(x)}{i!}$$

$\boxed{84}$ (a) $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

We show that for any "n", $f^{(n)}(0) = 0$

<u>First</u> $\lim\limits_{x\to 0} e^{-\frac{1}{x^2}} = \lim\limits_{x\to 0} \frac{1}{e^{\frac{1}{x^2}}} \underset{t=\frac{1}{x}}{=} \lim\limits_{t\to\infty} \frac{1}{e^{t^2}} = 0$ f is Cont.

$$f'(0) = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{x\to 0} \frac{1}{x e^{\frac{1}{x^2}}} \underset{t=\frac{1}{x}}{=} \lim_{t\to\infty} \frac{1}{\frac{1}{t} e^{t^2}} = \lim_{t\to\infty} \frac{t}{e^{t^2}}$$

$$\underset{\text{Hopital rule}}{=} \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0$$

Prove by <u>induction</u> on n, $\lim\limits_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0$

Assume $\lim\limits_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0$

$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^{n+1}} \underset{t=\frac{1}{x}}{=} \lim_{t\to\infty} \frac{1}{\frac{1}{t^{n+1}} \cdot e^{t^2}} = \lim_{t\to\infty} \frac{t^{n+1}}{e^{t^2}} = \lim_{t\to\infty} \frac{(n+1)t^n}{2te^{t^2}}$$

$$= \lim_{t\to\infty} \frac{(n+1)t^{n-1}}{2e^{t^2}} = \lim_{x\to 0} \frac{(n+1)}{2} \cdot \frac{e^{-\frac{1}{x^2}}}{x^{n-1}} = 0 \quad ✓$$

<u>Second</u> By indunction on "n" we prove $\frac{d^n e^{-\frac{1}{x^2}}}{dx} = P_n(\frac{1}{x}) e^{-\frac{1}{x^2}}$

where P_n is a polynomial & $x \neq 0$

$$\frac{d^{(n+1)} e^{-\frac{1}{x^2}}}{dx} = \frac{d(P_n(\frac{1}{x}) e^{-\frac{1}{x^2}})}{dx} = P_n'(\tfrac{1}{x}) \cdot (-\tfrac{1}{x^2}) e^{-\frac{1}{x^2}} + P_n(\tfrac{1}{x}) \cdot 2x^{-3} e^{-\frac{1}{x^2}}$$

$$= \left(-\frac{1}{x^2} P_n'(\tfrac{1}{x}) + \frac{2}{x^3} P_n(\tfrac{1}{x})\right) e^{-\frac{1}{x^2}}$$

$$\rightsquigarrow \boxed{P_{n+1}(x) = -x^2 P_n'(x) + 2x^3 P_n(x)}$$

Therefore, $f^{(n+1)}(0) = \lim_{x \to 0} \dfrac{f^{(n)}(x) - f^{(n)}(0)}{x}$

$$= \lim_{x \to 0} \frac{P_n(\frac{1}{x})\, e^{-1/x^2}}{x} = \lim_{x \to 0} \left(\frac{1}{x} P_n\left(\frac{1}{x}\right)\right) e^{-\frac{1}{x^2}} = 0$$

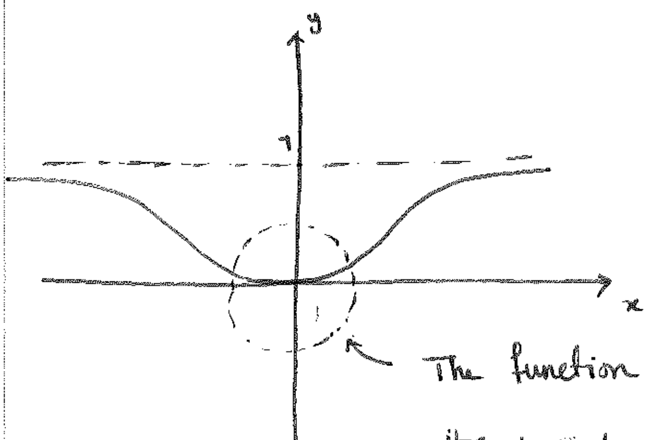
if $f^{(n)}(0) = 0$

since $\lim_{x \to 0} \left(\frac{1}{x}\right)^n e^{-\frac{1}{x^2}} = 0$

$\leadsto$ By induction on n we have $f^{(n)}(0) = 0$

Thus Maclaurin Series of $f(x)$ is equal to zero. but the function is nonzero.

(b)



The function $f(x)$ is very flat at origin, all of its derivatives at origin is zero.

85 (a) $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n \leadsto g'(x) = \sum_{n=0}^{\infty} \binom{k}{n} n x^{n-1} \qquad |x| < 1$

$$\leadsto g'(x) = \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} \cdot n\, x^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{(n-1)!} x^{n-1} = \sum_{n=1}^{\infty} k \binom{k-1}{n-1} x^{n-1}$$

$$\leadsto (1+x)g'(x) = k\left[\sum_{n=1}^{\infty} \binom{k-1}{n-1} x^n + \sum_{n=1}^{\infty} \binom{k-1}{n-1} x^{n-1}\right]$$

$$= k\left[1 + \sum_{n=1}^{\infty} \left(\binom{k-1}{n-1} + \binom{k-1}{n}\right) x^n\right]$$

$$\binom{k-1}{n-1} + \binom{k-1}{n} = \frac{(k-1)(k-2)\cdots(k-n+1)}{(n-1)!} + \frac{(k-1)(k-2)\cdots(k-n)}{n!}$$

$$= \frac{(k-1)\cdots(k-n+1)n + (k-1)(k-2)\cdots(k-n)}{n!}$$

$$= \frac{(k-1)\cdots(k-n+1)\left(n + k - n\right)}{n!} = \frac{k(k-1)\cdots(k-n+1)}{n!} = \binom{k}{n}$$

$$\rightsquigarrow (1+x)\, g'(x) = k \sum_{n=0}^{\infty} \binom{k}{n} x^n = k\, g(x)$$

(b) $h(x) = (1+x)^{-k} g(x) \rightsquigarrow h'(x) = -k(1+x)^{-k-1} g(x) + (1+x)^{-k} g'(x)$

$$= -\frac{k g(x)}{(1+x)^{k+1}} + \frac{g'(x)}{(1+x)^k} = \frac{g'(x)(1+x) - k g(x)}{(1+x)^{k+1}}$$

$$= 0$$

(c) $h'(x) = 0 \rightsquigarrow h(x) = C$ ← Constant

$$\rightsquigarrow \frac{g(x)}{(1+x)^k} = C \rightsquigarrow g(x) = C(1+x)^k$$

$$g(0) = 1 \rightsquigarrow C = 1 \rightsquigarrow g(x) = (1+x)^k$$