Section 9.5

$\boxed{6}$ $y' - y = e^x$ linear and $p(x) = -1$, $\int p(x)\,dx = -x + C$

$I(x) = e^{\int p(x)dx} = e^{-x}$ $\rightsquigarrow e^{-x}y' - e^{-x}y = 1$

$\rightsquigarrow (e^{-x}y)' = 1$

$\rightsquigarrow \int (e^{-x}y)'\,dx = \int 1\,dx$

$\rightsquigarrow e^{-x}y = x + C \rightsquigarrow y = e^x x + Ce^x$

$\boxed{8}$ $\underbrace{4x^3y + x^4y'}_{(x^4y)'} = \sin^3 x$ linear

$\rightsquigarrow \int (x^4y)'\,dx = \int \sin^3 x\,dx = \frac{\cos^3 x}{3} - \cos x + C$

$\rightsquigarrow x^4y = \frac{\cos^3 x}{3} - \cos x + C$

$\rightsquigarrow y = \frac{\cos^3 x}{3x^4} - \frac{\cos x}{x^4} + \frac{C}{x^4}$

$\boxed{10}$ $2xy' + y = 2\sqrt{x}$ $\rightsquigarrow y' + \frac{y}{2x} = \frac{1}{\sqrt{x}}$ linear and $p(x) = \frac{1}{2x}$

$I(x) = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln|x|} = \sqrt{x}$

$\rightsquigarrow$ ~~[illegible]~~ $\sqrt{x}\left(y' + \frac{y}{2x}\right) = 1$ $\rightsquigarrow \underbrace{\sqrt{x}\,y' + \frac{y}{2\sqrt{x}}}_{(\sqrt{x}\,y)'} = 1$

$\rightsquigarrow \int (\sqrt{x}\,y)'\,dx = \int 1\,dx$ $\rightsquigarrow \sqrt{x}\,y = x + C$

$\rightsquigarrow y = \sqrt{x} + \frac{C}{\sqrt{x}}$

$\boxed{16}$ $t^3 \frac{dy}{dt} + 3t^2 y = \cos t \qquad y(\pi) = 0$

linear equation and $t^3 \frac{dy}{dt} + 3t^2 y = (t^3 y)'$

$\rightsquigarrow \int (t^3 y)' \, dt = \int (\cos t) \, dt$

$\rightsquigarrow t^3 y = \sin t + C \rightsquigarrow y = \frac{\sin t}{t^3} + \frac{C}{t^3}$

$\rightsquigarrow y(\pi) = \frac{C}{\pi^3} = 0 \rightsquigarrow C = 0$

$\rightsquigarrow \boxed{y = \frac{\sin t}{t^3}}$

$\boxed{20}$ $(x^2+1) \frac{dy}{dx} + 3x(y-1) = 0 \qquad y(0) = 2$

$(x^2+1) \frac{dy}{dx} + 3xy = 3x \rightsquigarrow \frac{dy}{dx} + \frac{3x}{x^2+1} y = \frac{3x}{x^2+1}$ linear $P(x) = \frac{3x}{x^2+1}$

$\int P(x)\,dx = \int \frac{3x}{x^2+1} \, dx \underset{\substack{u = x^2+1 \\ du = 2x\,dx}}{=} \int \frac{3\,du}{2u} = \frac{3}{2} \ln|u| + C = \frac{3}{2} \ln(x^2+1) + C$

$\rightsquigarrow I(x) = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{3/2}$

$\rightsquigarrow (x^2+1)^{3/2} \left(\frac{dy}{dx} + \frac{3x}{x^2+1} y \right) = (x^2+1)^{3/2} \cdot \frac{3x}{x^2+1}$

$\rightsquigarrow \underbrace{(x^2+1)^{3/2} \frac{dy}{dx} + 3x\sqrt{x^2+1}\, y}_{\left((x^2+1)^{3/2} y\right)'} = 3x\sqrt{x^2+1} \rightsquigarrow \int \left((x^2+1)^{3/2} y\right)' dx = \int 3x\sqrt{x^2+1} \, dx$

$\rightsquigarrow (x^2+1)^{3/2} y = \int 3x\sqrt{x^2+1} \, dx \underset{\substack{u = x^2+1 \\ du = 2x\,dx}}{=} \int \frac{3}{2} \sqrt{u} \, du = u^{3/2} + C = (x^2+1)^{3/2} + C$

$\rightsquigarrow y = 1 + \frac{C}{(x^2+1)\sqrt{x^2+1}}$

$y(0) = 1 + C = 2 \rightsquigarrow \boxed{C = 1}$

$\boxed{23}$ $\dfrac{dy}{dx} + P(x)y = Q(x)\,y^n$ substitute $u = y^{1-n}$

$$\frac{du}{dx} = (1-n)\,y^{-n}\frac{dy}{dx} \rightsquigarrow y^{-n}\cdot\frac{dy}{dx} = \frac{1}{1-n}\,\frac{du}{dx}$$

$$\frac{dy}{dx}\cdot y^{-n} + P(x)\,y^{1-n} = Q(x) \rightsquigarrow \frac{1}{1-n}\,\frac{du}{dx} + P(x)\,u = Q(x)$$

$$\rightsquigarrow \frac{du}{dx} + (1-n)\,P(x)u = (1-n)\,Q(x)$$

$\boxed{24}$ $xy' + y = -xy^2 \rightsquigarrow y' + \frac{1}{x}y = -y^2$ substitute $u = y^{-1}$

$$\frac{du}{dx} = -y^{-2}y' \rightsquigarrow y^{-2}y' = -u'$$

$$\rightsquigarrow y^{-2}y' + \frac{1}{xy} = -1 \rightsquigarrow -u' + \frac{u}{x} = -1$$

$$\rightsquigarrow u' - \frac{u}{x} = 1$$

$$P(x) = -\frac{1}{x} \rightsquigarrow I(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\rightsquigarrow \underbrace{\frac{1}{x}u' - \frac{u}{x^2}}_{\left(\frac{u}{x}\right)'} = \frac{1}{x} \rightsquigarrow \int\left(\frac{u}{x}\right)'dx = \int \frac{1}{x}\,dx$$

$$\rightsquigarrow \frac{u}{x} = \ln|x| + C$$

$$\rightsquigarrow u = x\ln|x| + Cx$$

$$\rightsquigarrow \frac{1}{y} = x\ln|x| + Cx \rightsquigarrow y = \frac{1}{x\ln|x| + Cx}$$

$\boxed{26}$ $xy'' + 2y' = 12x^2$ Substitute $u = y'$

$\rightsquigarrow xu' + 2u = 12x^2 \rightsquigarrow u' + \frac{2}{x}u = 12x$ linear $P(x) = \frac{2}{x}$

$$I(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$$

$$\rightsquigarrow x^2\left(u' + \frac{2}{x}u\right) = 12x^3 \rightsquigarrow \underbrace{x^2u' + 2xu}_{(x^2u)'} = 12x^3 \rightsquigarrow x^2u = \int 12x^3\,dx$$

$$= 3x^4 + C$$

$$u = 3x^2 + \frac{c}{x^2} \rightsquigarrow y' = 3x^2 + \frac{c}{x^2}$$

$$\rightsquigarrow y = \int \left(3x^2 + \frac{c}{x^2}\right) dx = x^3 \boxed{- \frac{c}{x} + D}$$