Homework 2

Section 7.1

(9) $$\int \cos^{-1}x\,dx = x\cos^{-1}x - \int -\frac{x}{\sqrt{1-x^2}}\,dx = x\cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$w = 1-x^2 \rightsquigarrow dw = -2x\,dx$

$u = \cos^{-1}x \qquad dv = dx$

$du = -\dfrac{1}{\sqrt{1-x^2}} \qquad v = x$

$$= x\cos^{-1}x + \int \frac{-1}{2\sqrt{w}}\,dw = x\cos^{-1}x - \frac{1}{2}\underbrace{\int w^{-1/2}\,dw}_{-\frac{1}{2}\cdot 2\sqrt{w} = -\sqrt{1-x^2}} = \boxed{x\cos^{-1}x - \sqrt{1-x^2} + C}$$

(20) $$\int x\tan^2x\,dx = x(\tan x - x) - \int (\tan x - x)\,dx = x\tan x - x^2 - \ln|\sec x| + \frac{x^2}{2} + C$$
$$= \boxed{x\tan x - \ln|\sec x| - \frac{x^2}{2} + C}$$

$u = x \qquad dv = \tan^2x\,dx$

$du = 1 \qquad = (\sec^2x - 1)\,dx \rightsquigarrow v = \tan x - x$

(29) $$\int_0^{\pi} x\sin x\cos x\,dx = x\frac{\sin^2x}{2}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin^2x}{2}\,dx = 0 - \frac{1}{4}(x - \cos x\sin x)\Big|_0^{\pi}$$
$$= -\frac{1}{4}\left(\pi - 0 - (0-0)\right) = \boxed{-\pi/4}$$

$u = x \qquad dv = \sin x\cos x\,dx$

$du = 1 \qquad v = \int \sin x\cos x\,dx = \int w\,dw = \frac{w^2}{2} + C = \frac{\sin^2x}{2} + C$

$w = \sin x$

$dw = \cos x\,dx$

$\rightsquigarrow$ Set $v = \dfrac{\sin^2x}{2}$

$$\int \sin^2x\,dx = -\frac{1}{2}\cos x\sin x + \frac{1}{2}\int dx = \frac{1}{2}(x - \cos x\sin x) + C$$ (using the reductive formula)

You can also compute it using trig. identity : $\sin^2x = \dfrac{1-\cos 2x}{2}$

(40) $\int_0^{\pi} e^{\cos t} \sin 2t \, dt = \int_0^{\pi} 2e^{\cos t} \sin t \cos t \, dt = \int_1^{-1} -2e^{u} \cdot u \, du = \int_{-1}^{1} 2u e^{u} \, du$

$u = \cos t \rightsquigarrow du = -\sin t \, dt$

$u = \cos 0 = 1 \quad \cos \pi = -1$

$$= 2\left[u e^{u} \Big|_{-1}^{1} - \int_{-1}^{1} e^{u} du \right] = 2\left[e + e^{-1} - (e - e^{-1}) \right] = 4e^{-1}$$

(47) (a) $\int \sin^2 x \, dx = -\frac{1}{2} \cos x \sin x + \frac{2-1}{2} \int \sin^0 x \, dx = -\frac{1}{2} \underbrace{\sin x \cos x}_{\frac{1}{2}\sin 2x} + \frac{1}{2} x + C$

$= -\frac{1}{4} \sin 2x + \frac{1}{2} x + C$

(b) $\int \sin^4 x \, dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \int \sin^2 x \, dx$

$= -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \left(-\frac{1}{4} \sin 2x + \frac{1}{2} x \right) + C$

$= -\frac{1}{4} \cos x \sin^3 x - \frac{3}{16} \sin 2x + \frac{3}{8} x + C$

(48) (a) $\int \cos^n x \, dx$

$u = \cos^{n-1} x \qquad dv = \cos x \, dx \rightsquigarrow v = \sin x$

$du = (n-1) \cos^{n-2} x \cdot (-\sin x) dx = -(n-1) \sin x \cos^{n-2} x \, dx$

$= \sin x \cos^{n-1} x - \int \sin x \cdot \left(-(n-1) \sin x \cos^{n-2} x \, dx \right)$

$= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \underbrace{\sin^2 x}_{1 - \cos^2 x} dx$

$= \sin x \cos^{n-1} x - (n-1) \int \cos^n x \, dx + (n-1) \int \cos^{n-2} x \, dx$

$n \int \cos^n x \, dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx \rightsquigarrow \boxed{\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx}$

(b) $$\int \cos^2 x\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}\int \cancel{\cos^0} x\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$
$$= \frac{1}{4}\sin 2x + \frac{1}{2}x + C$$

(c) $$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left[\frac{1}{4}\sin 2x + \frac{1}{2}x\right] + C$$
$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{16}\sin 2x + \frac{3}{8}x + C$$

49 (a) $$\int_0^{\pi/2} \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x \Big|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$
$$= -[0-0] + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx \rightsquigarrow \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

(b) $$\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \frac{2}{3}(-\cos x)\Big|_0^{\pi/2} = \frac{2}{3}(0-(-1)) = \frac{2}{3}$$

$$\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$$

(c) $$\int_0^{\pi/2} \sin^{2n+1} x\,dx \overset{\text{use (a)}}{=} \frac{2n}{2n+1}\int_0^{\pi/2} \sin^{2n-1} x\,dx \overset{\text{use (a)}}{=} \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\int_0^{\pi/2} \sin^{2n-3} x\,dx$$
$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\int_0^{\pi/2} \sin^{2n-5} x\,dx$$

⋮ Repeat using formula in part (a)

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{2}{3}\underbrace{\int_0^{\pi/2} \sin x\,dx}_{=1}$$

$$= \frac{2\cdot4\cdot6\cdots2n}{3\cdot5\cdot7\cdots(2n+1)}$$

(50)

$$\int_0^{\pi/2} \sin^{2n} x \, dx \underset{\text{part a of ex. 49}}{=} \frac{2n-1}{2n} \int_0^{\pi/2} \sin^{2n-2} x \, dx = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \int_0^{\pi/2} \sin^{2n-4} x \, dx$$

repeat using formula in part (a) of ex. 49

$$= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \dots \cdot \frac{1}{2} \underbrace{\int_0^{\pi/2} \underset{= 1}{\sin^0 x} \, dx}_{\pi/2}$$

$$= \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} \cdot \frac{\pi}{2}$$

(74) (a)

$$\sin^{2n} x - \sin^{2n+1} x = \underset{\geq 0}{\sin^{2n} x} \, (\underset{\geq 0}{1 - \sin x}) \geq 0 \rightsquigarrow \sin^{2n} x \geq \sin^{2n+1} x$$

for any $0 \leq x \leq \pi/2$

$$\sin^{2n+1} x - \sin^{2n+2} x = \underset{\geq 0}{\sin^{2n+1} x} \, (\underset{\geq 0}{1 - \sin x}) \geq 0 \rightsquigarrow \sin^{2n+1} x \geq \sin^{2n+2} x$$

for any $0 \leq x \leq \pi/2$

$\rightsquigarrow$ for any $0 \leq x \leq \pi/2$ $\quad \sin^{2n} x \geq \sin^{2n+1} x \geq \sin^{2n+2} x$

$$\Downarrow$$

$$\int_0^{\pi/2} \sin^{2n} x \, dx \geq \int_0^{\pi/2} \sin^{2n+1} x \, dx \geq \int_0^{\pi/2} \sin^{2n+2} x \, dx$$

Therefore, $I_{2n} \geq I_{2n+1} \geq I_{2n+2}$.

(b)

$$I_{2n+2} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n+2)} \cdot \frac{\pi}{2}$$

$$I_{2n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} \cdot \pi/2$$

$$\rightsquigarrow \frac{I_{2n+2}}{I_{2n}} = \frac{\cancel{1 \cdot 3 \cdot 5 \cdots} (2n+1)}{\cancel{2 \cdot 4 \cdot 6 \cdots} (2n+2)} \cdot \cancel{\frac{\pi}{2}} \cdot \frac{\cancel{2 \cdot 4 \cdots 2n}}{\cancel{1 \cdot 3 \cdots (2n-1)}} \cdot \cancel{\frac{2}{\pi}}$$

$$= \boxed{\frac{2n+1}{2n+2}}$$

(c) $\dfrac{I_{2n+2}}{I_{2n}} \leq \dfrac{I_{2n+1}}{I_{2n}}$ (because $I_{2n+2} \leq I_{2n+1}$)

$\downarrow$

$\dfrac{2n+1}{2n+2} \leq \dfrac{I_{2n+1}}{I_{2n}}$ moreover, $I_{2n+1} \leq I_{2n} \rightsquigarrow \dfrac{I_{2n+1}}{I_{2n}} \leq 1 \rightsquigarrow \dfrac{2n+1}{2n+2} \leq \dfrac{I_{2n+1}}{I_{2n}} \leq 1$

$$\rightsquigarrow \underset{=1}{\underbrace{\lim_{n\to\infty} \frac{2n+1}{2n+2}}} \leq \lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} \leq \underset{1}{\underbrace{\lim_{n\to\infty} 1}} \rightsquigarrow \boxed{\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1}$$

(d) $I_{2n+1} = \dfrac{2\cdot 4\cdot \ldots \cdot 2n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1)}$ $\rightsquigarrow$ $\dfrac{I_{2n+1}}{I_{2n}} = \dfrac{2\cdot 2\cdot 4\cdot 4\cdot \ldots \cdot (2n)(2n)}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot \ldots \cdot (2n-1)(2n-1)(2n+1)} \cdot \dfrac{2}{\pi}$

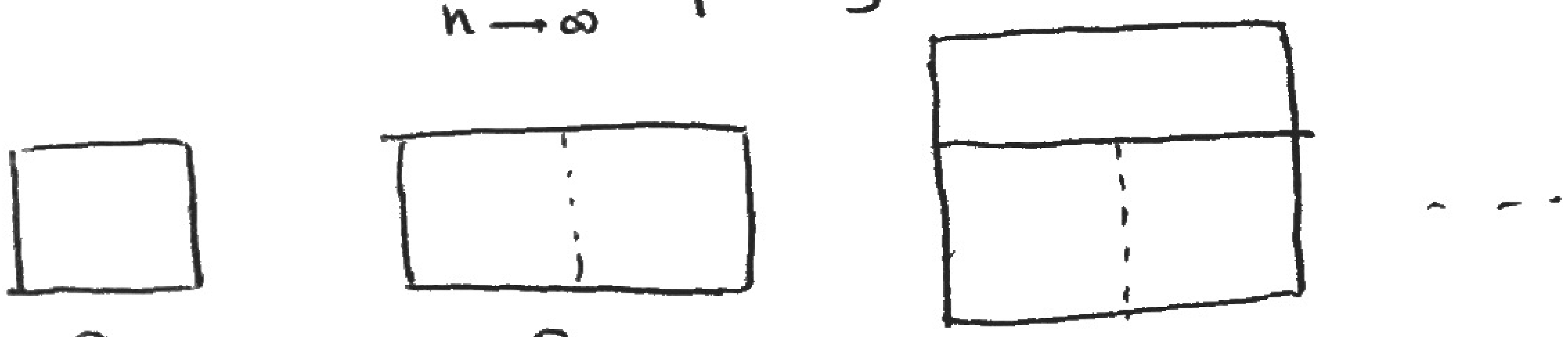
$I_{2n} = \dfrac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n} \cdot \pi/2$

$$= \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi}$$

$$\rightsquigarrow \lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \frac{2}{\pi} \lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = 1$$

$$\rightsquigarrow \boxed{\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}}$$

(e)



R_1 R_2 R_3 ...

Note that the area of the n-th rectangle R_n is equal to n. If n is even, $n = 2k$, changing R_{2k} to R_{2k+1}, width remains fixed and height multiplies by $\dfrac{2k+1}{2k}$ (because $\dfrac{\text{area}(R_{2k+1})}{\text{area}(R_{2k})} = \dfrac{2k+1}{2k}$). Moreover, if n is odd, $n = 2k+1$ changing R_{2k+1} to R_{2k+2}, height remains fixed & width multiplies by $\dfrac{2k+2}{2k+1}$.

So when $n = 2k$ the ratio of width to height multiplies with $\dfrac{2k}{2k+1}$ & when $n = 2k+1$ it multiplies with $\dfrac{2k+2}{2k+1}$. There fore by part it the limiting ratio is $\dfrac{\pi}{2}$.

Section 7.2

(17) $\int \sin^2 x \underbrace{\sin 2x}_{2\sin x \cos x}\, dx = 2\int \sin^3 x \cos x\, dx = 2\int u^3 du = \frac{u^4}{2} + C = \boxed{\frac{\sin^4 x}{2} + C}$

$u = \sin x$

$du = \cos x\, dx$

(49) $\int x \tan^2 x\, dx = \int x(\sec^2 x - 1)\, dx = \int x \sec^2 x\, dx - \int x\, dx = x\tan x - \int \tan x\, dx - \frac{x^2}{2} + C$

$u = x \quad dv = \sec^2 x\, dx$

$v = \tan x$

$= \boxed{x\tan x - \ln|\sec x| - \frac{x^2}{2} + C}$

(67) $\sin(-mx)\cos(-nx) = -\sin mx \cos nx \rightsquigarrow \sin mx \cos nx$ is odd.

$\rightsquigarrow \int_{-\pi}^{\pi} \sin(mx)\cos(nx) = 0$

(68) $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}\Big(\cos((m-n)x) - \cos((m+n)x)\Big)\, dx$

$= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right]_{-\pi}^{\pi} = 0$ because $\sin((m-n)\pi) = \sin((m+n)\pi) = 0$

when $m \neq n$

If $m = n$ $\quad \int_{-\pi}^{\pi} \frac{1}{2}\big(1 - \cos(2nx)\big)\, dx = \frac{x}{2} - \frac{\sin(2nx)}{4n}\Big|_{-\pi}^{\pi} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$

(69) $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}\big[\cos((m-n)x) + \cos((m+n)x)\big]\, dx \overset{m \neq n}{=}$

$\left(\frac{1}{2} \cdot \frac{\sin((m-n)x)}{m-n} + \frac{1}{2}\frac{\sin((m+n)x)}{m+n}\right)\Big|_{-\pi}^{\pi} = \boxed{0}$

If $m \neq n \rightsquigarrow \int_{-\pi}^{\pi} \frac{1}{2}[1 + \cos 2nx]\, dx = \frac{x}{2} + \frac{\sin 2nx}{4n}\Big|_{-\pi}^{\pi} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$

(70) $\left(f(x) = \sum_{n=1}^{N} a_n \sin(nx) = a_1 \sin x + a_2 \sin 2x + \cdots + a_N \sin Nx \right) \times \sin mx$

$$f(x)\sin(mx) = a_1 \sin x \sin(mx) + a_2 \sin(2x) \sin(mx) + \cdots + a_N \sin(Nx) \sin(mx)$$

$$\int_{-\pi}^{\pi} f(x) \sin(mx) = \int_{-\pi}^{\pi} \left[a_1 \sin x \sin(mx) + a_2 \sin(2x) \sin(mx) + \cdots + a_N \sin(Nx) \sin(mx) \right] dx$$

$$= \int_{-\pi}^{\pi} a_1 \sin x \sin(mx) dx + \int_{-\pi}^{\pi} a_2 \sin(2x) \sin(mx) dx + \cdots + \int_{-\pi}^{\pi} a_N \sin(Nx) \sin(mx) dx$$

$$\underset{\text{ex. 68}}{=} \int_{-\pi}^{\pi} a_m \sin(mx) \sin(mx) dx = a_m \pi$$

$$\leadsto \quad \boxed{a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) dx}$$