Homework 4

Section 7.8 :

(18) $$\int_2^\infty \frac{dv}{v^2+2v-3} = \int_2^\infty \frac{dv}{4(v-1)} + \int_2^\infty -\frac{dv}{4(v+3)} = \lim_{x\to\infty}\int_2^x \frac{dv}{4(v-1)} + \lim_{x\to\infty}\int_2^x -\frac{dv}{4(v+3)}$$

$v^2+2v-3=(v+3)(v-1)$

$\leadsto \dfrac{1}{v^2+2v-3} = \dfrac{A}{v+3} + \dfrac{B}{v-1}$

$\leadsto A(v-1)+B(v+3)=1 \leadsto \begin{matrix} A+B=0 \\ -A+3B=1 \end{matrix} \leadsto B=\frac{1}{4},\ A=-\frac{1}{4}$

$$= \lim_{x\to\infty}\left[\frac{1}{4}\ln(v-1)\right]_2^x + \lim_{x\to\infty}\left[-\frac{1}{4}\ln(v+3)\right]_2^x$$

$$= \lim_{x\to\infty}\left[\frac{1}{4}\ln(x-1) - \frac{1}{4}\overset{0}{\cancel{\ln 1}}\right] + \lim_{x\to\infty}\left[-\frac{1}{4}\ln(x+3)+\frac{1}{4}\ln 5\right]$$

$$= \lim_{x\to\infty}\left(\frac{1}{4}\ln\frac{x-1}{x+3}\right) + \frac{1}{4}\ln 5 = \boxed{\frac{\ln 5}{4}}$$

(30) $$\int_{-1}^2 \frac{x}{\underbrace{(x+1)^2}_{\text{has discontinuity at } x=-1}}\,dx = \lim_{t\to-1}\int_t^2 \frac{x}{(x+1)^2}\,dx = \lim_{t\to-1}\int_t^2\left(\frac{1}{x+1}-\frac{1}{(x+1)^2}\right)dx = \lim_{t\to-1}\left[\ln(x+1)+\frac{1}{x+1}\right]_t^2$$

$\left[\dfrac{x}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} \leadsto A(x+1)+B=x \leadsto A=1 \quad B=-1\right]$

$$= \lim_{t\to-1}\left[\ln 3+\frac{1}{3}-\ln(t+1)-\frac{1}{t+1}\right]$$

$$= \ln 3+\frac{1}{3} - \lim_{t\to-1}\frac{\overset{0}{(t+1)\ln(t+1)}+1}{t+1}$$

$$\lim_{t\to-1}(t+1)\ln(t+1) = \lim_{u\to0}\frac{\ln u}{\frac{1}{u}} \overset{\text{Hopital}}{=} \lim_{u\to0}\frac{\frac{1}{u}}{-\frac{1}{u^2}} = 0 \Rightarrow \underline{\text{diverges}}$$

(51) $$\int_1^\infty \frac{x+1}{\underbrace{\sqrt{x^4-x}}_{\text{discontinuity at } x=1}}\,dx$$

for $x>1$, $x^4-x<x^4 \leadsto \sqrt{x^4-x}<x^2$

$\leadsto \dfrac{x+1}{\sqrt{x^4-x}} > \dfrac{x+1}{x^2} > \dfrac{x}{x^2} = \dfrac{1}{x}$

$$\leadsto \int_2^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx > \underbrace{\int_2^\infty \frac{1}{x}\,dx}_{\text{diverges}} \leadsto \int_2^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx \text{ diverges} \leadsto \int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx : \text{diverges}$$

(52) $$\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$$

$2+e^x > e^x \leadsto \dfrac{\arctan x}{2+e^x} < \dfrac{\pi/2}{e^x}$

$0 \leq \arctan x \leq \frac{\pi}{2}$

$$\leadsto \int_0^\infty \frac{\arctan x}{2+e^x}\,dx < \underbrace{\int_0^\infty \frac{\pi}{2}e^{-x}\,dx}_{\text{Converges}} \leadsto \int_0^\infty \frac{\arctan x}{2+e^x}\,dx \text{ Converges}$$

(56) $$\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx = \int_2^3 \frac{1}{x\sqrt{x^2-4}}\,dx + \int_3^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$$

$x = 2\sec\theta$
$dx = 2\sec\theta\tan\theta\,d\theta$ $\Rightarrow \displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx = \int \frac{2\sec\theta\tan\theta}{2\sec\theta\, 2\tan\theta}\,d\theta = \int \frac{1}{2}\,d\theta = \frac{\theta}{2} + C = \frac{\sec^{-1}\frac{x}{2}}{2} + C$

$$= \underbrace{\left.\frac{\sec^{-1}\frac{x}{2}}{2}\right|_2^3}_{=\frac{\sec^{-1}\frac{3}{2}}{2}} + \lim_{x\to\infty} \frac{\sec^{-1}\frac{x}{2}}{2} - \frac{\sec^{-1}\frac{3}{2}}{2} = \lim_{x\to\infty}\frac{\sec^{-1}\frac{x}{2}}{2} = \boxed{\frac{\pi}{4}}$$

(80) $$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{c}{3x+1}\right)dx = \lim_{t\to\infty}\int_0^t \left(\frac{x}{x^2+1} - \frac{c}{3x+1}\right)dx = \lim_{t\to\infty} \frac{\ln(t^2+1)}{2} - \frac{c\ln(3t+1)}{3}$$

$$= \lim_{t\to\infty} \ln\sqrt{t^2+1} - \ln(3t+1)^{\frac{c}{3}} \quad (*)$$

$$\int \frac{x}{x^2+1} - \frac{c}{3x+1}\,dx = \underbrace{\int \frac{x}{x^2+1}\,dx}_{} - \underbrace{\int \frac{c}{3x+1}\,dx}_{}$$

$u = x^2+1$
$du = 2x\,dx$

$u = 3x+1$
$du = 3dx$

$$\int \frac{c}{3u}\,du = \frac{c}{3}\ln u + c' = \frac{c}{3}\ln(3x+1) + c'$$

$$\int \frac{du}{2u} = \frac{1}{2}\ln u + c = \frac{1}{2}\ln(x^2+1) + c$$

$$(*)\ \lim_{t\to\infty} \ln\sqrt{t^2+1} - \ln(3t+1)^{\frac{c}{3}} = \lim_{t\to\infty} \ln\left(\frac{\sqrt{t^2+1}}{(3t+1)^{\frac{c}{3}}}\right)$$

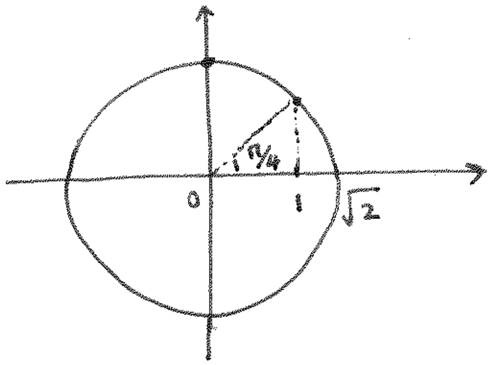
$$\frac{\sqrt{t^2+1}}{(3t+1)^{c/3}} = \frac{\sqrt{1+\frac{1}{t^2}}\cdot t}{(3+\frac{1}{t})^{c/3}\cdot t^{c/3}} = \frac{\sqrt{1+\frac{1}{t^2}}}{(3+\frac{1}{t})^{c/3}}\cdot t^{(1-c/3)} \rightsquigarrow \lim_{t\to\infty}\frac{\sqrt{t^2+1}}{(3t+1)^{\frac{c}{3}}} = \frac{1}{3^{c/3}}\cdot\lim_{t\to\infty} t^{1-\frac{c}{3}}$$

$\rightsquigarrow$
- if $1-c/3>0 \rightsquigarrow \lim_{t\to\infty} t^{1-c/3} = \infty \rightsquigarrow$ divergent
- if $1-c/3<0 \rightsquigarrow \lim_{t\to\infty} t^{1-c/3} = 0 \rightsquigarrow \lim_{t\to\infty} \ln\frac{\sqrt{t^2+1}}{(3t+1)^{c/3}} = -\infty \rightsquigarrow$ divergent
- if $1-\frac{c}{3}=0 \rightsquigarrow \boxed{c=3} \rightsquigarrow \lim_{t\to\infty}\ln\frac{\sqrt{t^2+1}}{3t+1} = \ln\frac{1}{3} = \boxed{-\ln 3}$

Section 8.1

2) $Y = \sqrt{2-x^2} \quad 0 \leq x \leq 1 \rightsquigarrow \dfrac{dy}{dx} = \dfrac{-2x}{2\sqrt{2-x^2}} = \dfrac{x}{\sqrt{2-x^2}}$

$r = \sqrt{2}$



$$\rightsquigarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{2-x^2}} = \sqrt{\frac{2}{2-x^2}} = \frac{\sqrt{2}}{\sqrt{2-x^2}}$$

$$\rightsquigarrow \int_0^1 \frac{\sqrt{2}}{\sqrt{2-x^2}}\,dx = \int_0^{\pi/4} \frac{2\cos\theta\, d\theta}{\sqrt{2}\cos\theta} = \int_0^{\pi/4} \sqrt{2}\, d\theta = \sqrt{2}\,\frac{\pi}{4} \boxed{= \frac{\pi\sqrt{2}}{4}}$$

$x = \sqrt{2}\sin\theta$

$dx = \sqrt{2}\cos\theta\, d\theta$

$\rightsquigarrow \begin{cases} \sqrt{2}\sin\theta = 0 \rightsquigarrow \theta = 0 \\ \sqrt{2}\sin\theta = 1 \rightsquigarrow \sin\theta = \pi/4 \end{cases}$

$$\text{arc length} = \frac{1}{8}\cdot\left(2\pi\cdot\sqrt{2}\right) = \boxed{\frac{\pi}{4}\cdot\sqrt{2}}$$

20) $y = 1 - e^{-x}, \ 0 \leq x \leq 2 \rightsquigarrow \dfrac{dy}{dx} = e^{-x} \rightsquigarrow \sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{1+e^{-2x}}$

$$\int_0^2 \sqrt{1+e^{-2x}}\,dx = (*)$$

$u = \sqrt{1+e^{-2x}} \rightsquigarrow du = \dfrac{-2e^{-2x}}{2\sqrt{1+e^{-2x}}}dx = -\dfrac{e^{-2x}}{\sqrt{1+e^{-2x}}}dx$

$\Updownarrow$

$u^2 = 1+e^{-2x} \quad \rightsquigarrow \dfrac{du}{1-u^2} = \dfrac{dx}{\sqrt{1+e^{-2x}}} \rightsquigarrow \dfrac{u^2}{1-u^2}du = \sqrt{1+e^{-2x}}\,dx$

$\Updownarrow$

$e^{-2x} = u^2 - 1$

$$(*) = \int_{\sqrt{2}}^{\sqrt{1+e^{-4}}} \frac{u^2}{1-u^2}\,du = \int_{\sqrt{2}}^{\sqrt{1+e^{-4}}} \left(\frac{1}{1-u^2} - 1\right) du = \int_{\sqrt{2}}^{\sqrt{1+e^{-4}}} \left(\frac{1}{2}\left(\frac{1}{1-u}+\frac{1}{1+u}\right) - 1\right) du$$

$$= \left[-\frac{1}{2}\ln(1-u) + \frac{1}{2}\ln(1+u) - u\right]_{\sqrt{2}}^{\sqrt{1+e^{-4}}} = -\frac{1}{2}\ln\frac{1-\sqrt{1+e^{-4}}}{1-\sqrt{2}} + \frac{1}{2}\ln\frac{1+\sqrt{1+e^{-4}}}{1+\sqrt{2}} + \sqrt{2} - \sqrt{1+e^{-4}}$$

(37) $y = \sin^{-1}x + \sqrt{1-x^2}$ starting at $(0,1)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{2x}{2\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} \rightsquigarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{(1-x)^2}{1-x^2}} = \sqrt{1+\frac{1-x}{1+x}}$$

$$= \sqrt{\frac{2}{1+x}}$$

$$\rightsquigarrow \int_0^x \sqrt{\frac{2}{1+t}}\,dt = \int_0^x \sqrt{2}\cdot\frac{1}{\sqrt{1+t}}\,dt = \sqrt{2}\cdot 2\sqrt{1+t}\,\Big|_0^x = \boxed{2\sqrt{2}\cdot\sqrt{1+x} - 2\sqrt{2}}$$

(45) $y = \int_1^x \sqrt{t^3-1}\,dt \rightsquigarrow \frac{dy}{dx} = \sqrt{x^3-1} \rightsquigarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+x^3-1} = \sqrt{x^3}$

$$\rightsquigarrow \int_1^4 \sqrt{x^3}\,dx = \frac{x^{5/2}}{5/2}\Big|_1^4 = \frac{2}{5}(32-1) = \boxed{\frac{62}{5}}$$