Homework 5 :

Section 8.2 :

(12) $y = \frac{x^3}{6} + \frac{1}{2x} \quad \frac{1}{2} \le x \le 1$

$$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$\rightsquigarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^4}{4} + \frac{1}{4x^4} - \frac{1}{2}$$

$$= \frac{1}{2} + \frac{x^4}{4} + \frac{1}{4x^4}$$

$$= \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

$$\rightsquigarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{x^2}{2} + \frac{1}{2x^2}$$

$$\text{Area} = \int_{1/2}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{1/2}^{1} 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$

$$= \int_{1/2}^{1} 2\pi \left[\frac{x^5}{12} + \frac{x}{4} + \frac{x}{12} + \frac{1}{4x^3}\right] dx$$

$$= 2\pi \left[\frac{x^6}{72} + \frac{x^2}{8} + \frac{x^2}{24} - \frac{1}{8x^2}\right]_{1/2}^{1}$$

$$= 2\pi \left[\frac{1}{72} + \cancel{\frac{1}{8}} + \frac{1}{24} - \cancel{\frac{1}{8}} - \left(\frac{1}{64.72} + \frac{1}{32} + \frac{1}{96} - \frac{1}{2}\right)\right] = \boxed{\frac{3966\pi}{4608}}$$

(16) $x^{2/3} + y^{2/3} = 1 \quad 0 \le y \le 1 \quad \rightsquigarrow x = (1 - y^{2/3})^{3/2} \rightsquigarrow \frac{dx}{dy} = \frac{3}{2}(1 - y^{2/3})^{1/2} \cdot \left(-\frac{2}{3} y^{-1/3}\right)$

$$= -(1 - y^{2/3})^{1/2} \cdot y^{-1/3}$$

$$\rightsquigarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left[(1 - y^{2/3}) \cdot y^{-2/3}\right]$$

$$= 1 + y^{-2/3} - 1 = y^{-2/3}$$

$$\rightsquigarrow ds = y^{-2/3}\, dy$$

$$\text{Area} = \int 2\pi x\, ds = \int_0^1 2\pi (1 + y^{2/3})^{3/2} \cdot y^{-2/3}\, dy$$

$$\left(u = y^{1/3} \rightsquigarrow du = \frac{1}{3} y^{-2/3}\, dy \quad \rightsquigarrow 3du = y^{-2/3}\, dy, \quad y = 0 \rightsquigarrow u = 0, \quad y = 1 \rightsquigarrow u = 1 \right)$$

$$= \int_0^1 2\pi (1 + u^2)^{3/2} \cdot 3du = \int_0^1 6\pi (1 + u^2)^{3/2}\, du$$

$u = \tan\theta \quad du = \sec^2\theta\, d\theta$

$u = 0 \rightsquigarrow \theta = 0$

$u = 1 \rightsquigarrow \theta = \pi/4$

$$= \int_0^{\pi/4} 6\pi \sec^3\theta \cdot \sec^2\theta\, d\theta$$

(27) $$\int_1^\infty 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^\infty 2\pi\cdot\frac{1}{x}\cdot\frac{\sqrt{1+x^4}}{x^2}\,dx = \int_1^\infty 2\pi\cdot\frac{\sqrt{1+x^4}}{x^3}\,dx$$

$$\left(\frac{dy}{dx} = -\frac{1}{x^2} \rightsquigarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{x^4}} = \frac{\sqrt{1+x^4}}{x^2}\right)$$

$$\sqrt{1+x^4} > \sqrt{x^4} = x^2 \rightsquigarrow \int_1^\infty 2\pi\,\frac{\sqrt{1+x^4}}{x^3}\,dx > \int_1^\infty 2\pi\,\frac{x^2}{x^3}\,dx = \underbrace{\int_1^\infty \frac{2\pi}{x}\,dx = \infty}_{\text{diverges}}$$

$$\rightsquigarrow \int_1^\infty 2\pi\,\frac{\sqrt{1+x^2}}{x^3}\,dx \overset{\text{diverges}}{=} \infty$$

(28) $$\int_0^\infty 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^\infty 2\pi e^{-x}\sqrt{1+(e^{-x})^2}\,dx \underset{\substack{u=e^{-x}\\ du=-e^{-x}dx}}{=} -\int_1^0 2\pi\sqrt{1+u^2}\,du$$

$$\left(y=e^{-x} \rightsquigarrow \frac{dy}{dx} = -e^{-x} \rightsquigarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+(e^{-x})^2}\right)$$

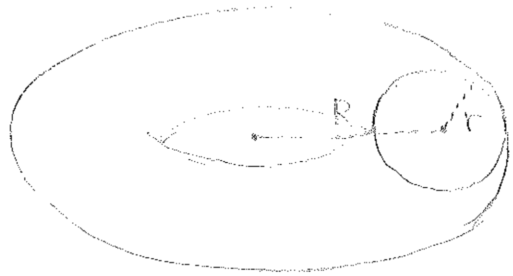
$$= \int_0^1 2\pi\sqrt{1+u^2}\,du = \int_0^{\pi/4} 2\pi\,\underbrace{\sec\theta\cdot\sec^2\theta}_{\sec^3\theta}\,d\theta \overset{\text{Ex 7.2.8}}{=} 2\pi\cdot\frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4}$$

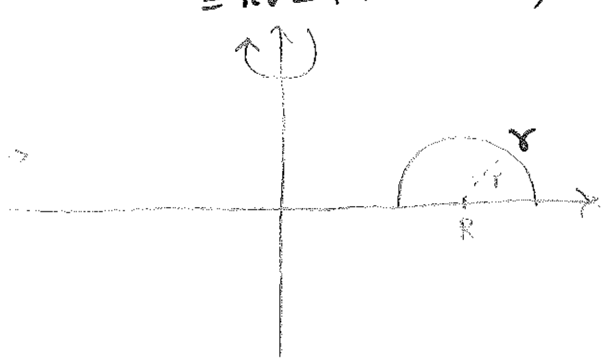
$u = \tan\theta$
$du = \sec^2\theta\,d\theta$

$$= \pi\cdot\left[\sqrt{2}\cdot 1 + \ln|\sqrt{2}+1|\right]$$

$$= \pi\sqrt{2} + \pi\ln(\sqrt{2}+1)$$

(32)





Area = 2 · (area of the surface obtained by rotating γ about y-axis)

$$\text{Area} = 2 \cdot \int 2\pi x \, ds$$

$$(x-R)^2 + y^2 = r^2 \rightsquigarrow y = \sqrt{r^2 - (x-R)^2}$$

$$\rightsquigarrow \frac{dy}{dx} = \frac{-\cancel{2}(x-R)}{\cancel{2}\sqrt{r^2-(x-R)^2}} = -\frac{x-R}{\sqrt{r^2-(x-R)^2}}$$

$$\rightsquigarrow ds = \sqrt{1 + \frac{(x-R)^2}{r^2-(x-R)^2}}\, dx = \sqrt{\frac{r^2}{r^2-(x-R)^2}}\, dx = \frac{r}{\sqrt{r^2-(x-R)^2}}\, dx$$

$$\text{Area} = 2 \cdot \int_{R-r}^{R+r} 2\pi x \cdot \frac{r}{\sqrt{r^2-(x-R)^2}}\, dx = 2 \cdot \int_{-r}^{r} \frac{2\pi (R+u) r}{\sqrt{r^2-u^2}}\, du = 2\int_{-r}^{r} \frac{2\pi R r}{\sqrt{r^2-u^2}}\, du + \underbrace{2\int_{-r}^{r} \frac{2\pi r u}{\sqrt{r^2-u^2}}\, du}_{=0}$$

$$\left(\begin{array}{l} u = x - R \\ du = dx \end{array} \right)$$

$\left(\text{because } \frac{2\pi r u}{\sqrt{r^2-u^2}} \text{ is } \underline{\underline{\text{odd}}}! \right)$

$$= 2\int_{-r}^{r} \frac{2\pi R r}{\sqrt{r^2-u^2}}\, du = 4\pi R r \int_{-r}^{r} \frac{1}{\sqrt{r^2-u^2}}\, du$$

$$\left(u = r\sin\theta \rightsquigarrow du = r\cos\theta\, d\theta \right)$$

$$= 4\pi R r \int_{-\pi/2}^{\pi/2} \frac{r\cancel{\cos}\theta\, d\theta}{r\cancel{\cos}\theta} = \boxed{4\pi^2 R r}$$

Section 11.1

(46) $a_n = 2^{-n}\cos n\pi = 2^{-n}\cdot(-1)^n = \frac{(-1)^n}{2^n}$

$-\frac{1}{2^n} \leq \frac{(-1)^n}{2^n} \leq \frac{1}{2^n}$, $\lim_{n\to\infty}\frac{1}{2^n} = 0 \Rightarrow \lim_{n\to\infty}\frac{(-1)^n}{2^n} = 0$ Converges

(48) $a_n = \sqrt[n]{n} = n^{\frac{1}{n}} = \left(e^{\ln n}\right)^{\frac{1}{n}} = e^{\frac{\ln n}{n}} \rightsquigarrow \{\sqrt[n]{n}\}$ Converges to "0".

$f(x) = \frac{\ln x}{x}$ $\quad \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1}{x} = 0$

(80) $a_{n+1} = \sqrt{2+a_n}$ and $a_1 = \sqrt{2}$

(a) • Use induction to show $\{a_n\}$ is increasing : $a_1 = \sqrt{2}$, $2+\sqrt{2} > 2$

$$\Downarrow$$

$$\underbrace{\sqrt{2+\sqrt{2}}}_{a_2} > \underbrace{\sqrt{2}}_{a_1}$$

$$\Downarrow$$

$$a_2 > a_1$$

Assume $a_{n-1} < a_n$, therefore $2+a_{n-1} < 2+a_n$

$\longrightarrow \sqrt{2+a_{n-1}} < \sqrt{2+a_n}$

$\longrightarrow a_n < a_{n+1}$ ✓ $\Rightarrow$ The seq. is increasing.

• Use induction to show $\{a_n\}$ is bounded by 3 $\quad a_1 = \sqrt{2} < 3$

If $a_{n-1} < 3 \rightsquigarrow \sqrt{2+a_{n-1}} < \sqrt{2+3} = \sqrt{5}$

$\rightsquigarrow a_n < \sqrt{5} < 3 \rightsquigarrow a_n < 3 \Rightarrow \{a_n\}$: bounded by 3

$\Rightarrow$ the seq. is Convergent.

(b) $a_{n+1} = \sqrt{2+a_n} \rightsquigarrow \lim_{n\to\infty} a_{n+1} = \sqrt{2+\lim_{n\to\infty} a_n}$

$\xrightarrow[L = \lim a_n]{} L = \sqrt{2+L} \rightsquigarrow L^2 - L - 2 = 0 \rightsquigarrow (L-2)(L+1) = 0$

for all n $\quad \rightsquigarrow \boxed{L=2}$

(82) $a_1 = 2 \qquad a_{n+1} = \dfrac{1}{3 - a_n}$

- Use induction to show $0 < a_n \leq 2$
  - first $0 < a_1 \leq 2$ ✓
  - Assume $0 < a_{n-1} \leq 2 \rightsquigarrow 3 > 3 - a_{n-1} \geq 1 \rightsquigarrow \dfrac{1}{3} < \dfrac{1}{3 - a_{n-1}} \leq 1$

$$\rightsquigarrow \frac{1}{3} < a_n \leq 1$$

$$\rightsquigarrow 0 < a_n \leq 2 \ ✓$$

- Use induction to show the Seq. is decreasing:
  - $a_1 = 2 \quad a_2 = 1 \rightsquigarrow a_2 < a_1$ ✓
  - Assume $a_{n-1} > a_n \rightsquigarrow 3 - a_{n-1} < 3 - a_n \rightsquigarrow \dfrac{1}{3 - a_{n-1}} > \dfrac{1}{3 - a_n}$

$$\rightsquigarrow a_n > a_{n+1} \ ✓$$

$\Rightarrow$ Seq. is decreasing & bounded

$$L = \lim_{n \to \infty} a_n \rightsquigarrow L = \frac{1}{3 - L} \rightsquigarrow 3L - L^2 = 1 \rightsquigarrow L^2 - 3L + 1 = 0$$

$$\rightsquigarrow L = \frac{3 + \sqrt{5}}{2} \text{ or } \frac{3 - \sqrt{5}}{2}$$

$$\xrightarrow[\text{for all } n]{\text{since } a_n \leq 2} \boxed{L = \frac{3 - \sqrt{5}}{2}}$$

(83) (a) f_n = # pairs of rabbits in the nth month

Note that the pairs of rabits that where produced in or before the $(n-2)$th month, Can produce a new pair in the nth month. So f_{n-2} pairs will be added to the pairs that are already produced which is f_{n-1}. Therefore: $f_n = f_{n-1} + f_{n-2}$

(b) $a_n = \dfrac{f_{n+1}}{f_n} = \dfrac{f_n + f_{n-1}}{f_n} = 1 + \dfrac{f_{n-1}}{f_n} = 1 + \dfrac{1}{\frac{f_n}{f_{n-1}}} = 1 + \dfrac{1}{a_{n-1}}$

$\rightsquigarrow$ If $L = \lim a_n \Rightarrow L = 1 + \dfrac{1}{L} \rightsquigarrow L^2 - L - 1 = 0 \rightsquigarrow L = \frac{3}{2}$ ✓

92 (a) $\lim_{n\to\infty} a_{2n} = L$ & $\lim_{n\to\infty} a_{2n+1} = L$

def : for $\varepsilon > 0$, there exist $N > 0$ s.t. for $n > N$, $L-\varepsilon < a_{2n} < L+\varepsilon$

there exist $M > 0$ s.t. for $n > M$, $L-\varepsilon < a_{2n+1} < L+\varepsilon$

$\Longrightarrow$ for $\tilde{N} = \max\{2N, 2M+1\}$ we have for any $n > \tilde{N}$,

$L-\varepsilon < a_n < L+\varepsilon$

$\Longrightarrow$ $\{a_n\}$ is Convergent and $\lim_{n\to\infty} a_n = L$

(b) $a_1 = 1$

$a_2 = 1 + \frac{1}{1+1} = \frac{3}{2}$

$a_3 = 1 + \frac{1}{1+\frac{3}{2}} = 1 + \frac{2}{5} = \frac{7}{5}$

$a_4 = 1 + \frac{1}{1+\frac{7}{5}} = 1 + \frac{5}{12} = \frac{17}{12}$

$a_5 = 1 + \frac{1}{1+\frac{17}{12}} = 1 + \frac{12}{29} = \frac{41}{29}$

$a_6 = 1 + \frac{1}{1+\frac{41}{29}} = 1 + \frac{29}{70} = \frac{99}{70}$

$a_7 = 1 + \frac{1}{1+\frac{99}{70}} = 1 + \frac{70}{169} = \frac{239}{169}$

$a_8 = 1 + \frac{1}{1+\frac{239}{169}} = 1 + \frac{169}{408} = \frac{577}{408}$

We want to show that $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are Convergent.

First $a_{n+1} = 1 + \frac{1}{1+a_n}$ since $a_n > 0$

$\rightsquigarrow \frac{1}{1+a_n} < 1 \rightsquigarrow a_{n+1} < 2$

$\rightsquigarrow 1 < a_n < 2$ bounded ✓

$\rightsquigarrow$ both $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are bounded.

Second Use induction to show $\{a_{2n+1}\}$ is increasing.

$a_1 < a_3$ ✓

Assume $a_{2n-1} < a_{2n+1} \rightsquigarrow 1+a_{2n-1} < 1+a_{2n+1}$

$\rightsquigarrow \frac{1}{1+a_{2n-1}} > \frac{1}{1+a_{2n+1}}$

$\rightsquigarrow 1 + \frac{1}{1+a_{2n-1}} > \frac{1}{1+a_{2n+1}} \rightsquigarrow a_{2n} > a_{2n+2}$

$\rightsquigarrow \frac{1}{1+a_{2n}} < \frac{1}{1+a_{2n+2}} \rightsquigarrow a_{2n+1} < a_{2n+3}$ ✓

Third, $\{a_{2n}\}$ is decreasing because $a_{2n} = 1+\dfrac{1}{1+a_{2n-1}} > 1+\dfrac{1}{1+a_{2n+1}} = a_{2n+2}$

because $a_{2n-1} < a_{2n+1}$

Therefore, both $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are Convergent.

Assume $L = \lim_{n\to\infty} a_{2n} \rightsquigarrow$ since $a_{2n} = 1+\dfrac{1}{1+\dfrac{1}{1+a_{2n-1}}} \rightsquigarrow L = 1+\dfrac{1}{1+\dfrac{1}{L}}$

$$= 1+\frac{L}{L+1} = \frac{2L+1}{L+1}$$

$$\rightsquigarrow L^2+L = 2L+1$$

$$\rightsquigarrow L^2-L+1 = 0 \rightsquigarrow L = \frac{1}{2} \text{ or } \frac{3}{2}$$

since $1 < a_n < 2$ we have $\boxed{L = \frac{3}{2}}$

In a similar way, you can show that $\lim_{n\to\infty} a_{2n+1} = \frac{3}{2}$

Therefore, $\boxed{\lim_{n\to\infty} a_n = \frac{3}{2}}$