Homework 7

Section 11.2

(16) (a) $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} a_j$ are not different. only the indices

i and j are different but both of them are equal to

$a_1 + \cdots + a_n$.

(b) $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$

$$\sum_{i=1}^{n} a_j = \overbrace{a_j + a_j + \cdots + a_j}^{n\text{-times}} = n \cdot a_j$$

(24) $$\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n} = \sum_{n=0}^{\infty} \frac{3^2}{(-2)} \cdot \frac{3^{n-1}}{(-2)^{n-1}} = \sum_{n=0}^{\infty} -\frac{9}{2} \cdot \left(-\frac{3}{2}\right)^{n-1}$$

$\rightsquigarrow$ geometric series with $a = -\frac{9}{2}$ and $r = -\frac{3}{2}$

Since $\left|-\frac{3}{2}\right| > 1$, the series is divergent.

(38) $$\sum_{k=0}^{\infty} (\sqrt{2})^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^k \rightsquigarrow \text{geometric series with } a=1 \quad r = \frac{1}{\sqrt{2}}$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^k = \left(1 + \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^2 + \cdots\right)$$

Since $\frac{1}{\sqrt{2}} < 1$ the series is convergent. The sum is equal to

$$\frac{a}{1-r} = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1}$$

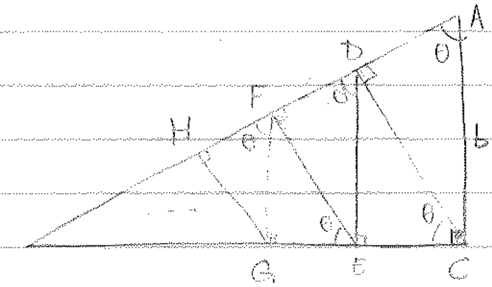
(68) $S_n = 3 - n2^{-n} \rightsquigarrow a_1 + a_2 + \cdots + a_n = 3 - n2^{-n}$

$$- \quad a_1 + a_2 + \cdots + a_{n-1} = 3 - (n-1)\,2^{-(n-1)}$$

$$a_n = -n2^{-n} + (n-1)2^{-(n-1)} = \frac{n-1}{2^{n-1}} - \frac{n}{2^n} = \frac{n-2}{2^n}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(3 - \frac{n}{2^n}\right) = 3 - \lim_{n\to\infty}\frac{n}{2^n} = 3 - \lim_{n\to\infty}\frac{1}{2^n \ln 2} = 3$$

(80)



$\frac{|CD|}{|AC|} = \sin\theta \rightsquigarrow |CD| = b\sin\theta$

$|DE| = |CD| \cdot \sin\theta = b\sin^2\theta$

$|EF| = |DE| \cdot \sin\theta = b\sin^3\theta$

$|FG| = |EF| \cdot \sin\theta = b\sin^4\theta$

$\vdots$

$\rightsquigarrow \text{Sum} = \sum_{n=1}^{\infty} b\sin^n\theta \leftsquigarrow$ geometric series with $a = b\sin\theta$, $r = \sin\theta$

since $|\sin\theta| < 1 \rightsquigarrow \boxed{\text{Sum} = \frac{b\sin\theta}{1-\sin\theta}}$

Section 11.3

(18) $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2+1} = \sum_{n=2}^{\infty} \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1}$, $f'(x) = -\frac{2x}{(x^2+1)^2} < 0$ for $x \geq 2$, also $f(x) > 0$.

$\rightsquigarrow \int_2^{\infty} \frac{1}{x^2+1}\,dx = \int_{\tan^{-1}2}^{\pi/4} \frac{1}{\sec^2\theta} \cdot \sec^2\theta\, d\theta = \theta \Big|_{\tan^{-1}2}^{\pi/4} = \frac{\pi}{4} - \tan^{-1}2$

$x = \tan\theta$
$dx = \sec^2\theta\, d\theta$

$\rightsquigarrow \sum_{n=2}^{\infty} \frac{1}{n^2+1}$ is Convergent.

(22) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

$f(x) = \frac{\ln x}{x^2} \rightsquigarrow f'(x) = \frac{\frac{1}{x}\cdot x^2 - 2x\ln x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4}$

$x \geq 2 \rightsquigarrow \ln x > \frac{1}{2} \rightsquigarrow 1 - 2\ln x < 0 \rightsquigarrow f'(x) < 0$

Moreover, for $x \geq 2$, $f(x) \geq 0$ therefore, we can use integral test.

$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$

$u = \ln x$ $dV = \frac{dx}{x^2} \rightsquigarrow du = \frac{1}{x}dx$, $v = -\frac{1}{x}$

Thus, $\displaystyle\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln t}{t} - \frac{1}{t} + \frac{\ln 2}{2} + \frac{1}{2}\right]$

$$= \frac{\ln 2}{2} + \frac{1}{2}$$

$\rightsquigarrow \displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ is Convergent.

(28) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{1+n^2}$

$f(x) = \dfrac{\cos^2 x}{1+x^2} \rightsquigarrow f'(x) = \dfrac{2\cos x\,(-\sin x)(1+x^2) - 2x\cos^2 x}{(1+x^2)^2}$

$$= -\left(\frac{\sin(2x)(1+x^2) + 2x\cos^2 x}{(1+x^2)^2}\right)$$

$f'(x)$ is not negative for $x \geq a$, for a Constant a. For example,

$f'\left((4n+3)\cdot\frac{\pi}{4}\right) > 0$ for any $n \geq 1$

(32) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$: Convergent for $p>1$

$f(x) = \dfrac{\ln x}{x^p} \geq 0$ for $x \geq 1$ $\quad f'(x) = \dfrac{\frac{1}{x}\cdot x^p - p x^{p-1}\ln x}{x^{2p}} = \dfrac{x^{p-1}(1-p\ln x)}{x^{2p}}$

$$= \frac{1-p\ln x}{x^{p+1}} \rightsquigarrow \text{for } x \geq e^{1/p}$$

$f(x)$ is decreasing.

Use integral test $\rightsquigarrow$ $\displaystyle\int \frac{\ln x}{x^p}\,dx = \frac{x^{-p+1}}{-p+1}\cdot \ln x + \int \frac{x^{-p+1}}{p-1}\cdot\frac{1}{x}\,dx = -\frac{x^{-p+1}}{p-1}\ln x - \frac{x^{-p+1}}{(p-1)^2} + C$

$u = \ln x \quad du = \frac{1}{x}dx$

$\rightsquigarrow$

$dv = x^{-p}dx \quad v = \dfrac{x^{-p+1}}{-p+1}$

$\rightsquigarrow \displaystyle\int_1^{\infty} \frac{\ln x}{x^p}\,dx = \lim_{t\to\infty}\left[-\frac{t^{-p+1}}{p-1}\ln t - \frac{t^{-p+1}}{(p-1)^2} + \frac{1}{(p-1)^2}\right]$

for $p-1>0$, $\displaystyle\lim_{t\to\infty} t^{-p+1} = 0$ and $\displaystyle\lim_{t\to\infty} \frac{\ln t}{t^{p-1}} = \lim_{t\to\infty} \frac{\frac{1}{t}}{(p-1)t^{p-2}} = 0$: Convergent!

for $p<1$ $\displaystyle\lim_{t\to\infty} t^{-p+1} = \infty$ divergent!

(33) $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ Convergent for $x > 1 \rightsquigarrow$ domain ζ is $(1,\infty)$.

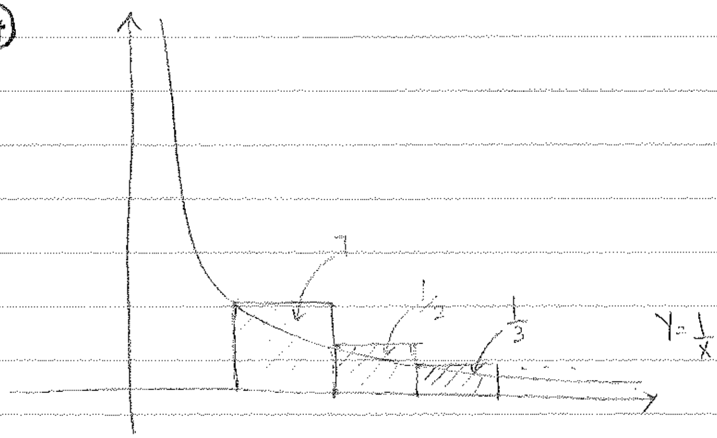
(34) (a) $\sum_{n=2}^{\infty} \frac{1}{n^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) - 1 = \boxed{\frac{\pi^2}{6} - 1}$

(b) $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=4}^{\infty} \frac{1}{n^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) - \overbrace{\left(1 + \frac{1}{4} + \frac{1}{9}\right)}^{49/36}$

$= \boxed{\frac{\pi^2}{6} - \frac{49}{36}}$

(c) $\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} = \boxed{\frac{\pi^2}{24}}$

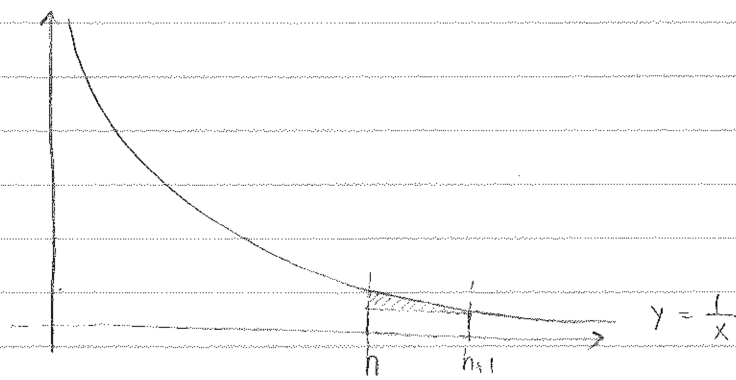
(44)



(a) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^n \frac{1}{x}\,dx = \ln n - \ln 1 = \ln n$

$\rightsquigarrow t_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n > 0$

(b)



Area under the curve $\frac{1}{x}$ from $x = n$ to $x = n+1$ is equal to $\int_n^{n+1} \frac{1}{x}\,dx$

$= \ln(n+1) - \ln(n)$

Therefore, $t_n - t_{n+1}$ is equal to the area of the region bounded between $y = \frac{1}{x}$, $y = \frac{1}{n+1}$ and $x = n$. $\Rightarrow$ $t_n - t_{n+1} > 0$ $\Rightarrow \{t_n\}$: decreasing

ⓒ $\{t_n\}$ is a decreasing sequence with positive terms. Therefor for any n $0 < t_n \leqslant t_1$ $\Rightarrow \{t_n\}$ is bounded and monotone.

Thus, it is Convergent.