Homework 7

$\boxed{11.4}$

(8) $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$ $\qquad 5^n-1 < 5^n \rightsquigarrow \frac{6^n}{5^n-1} > \frac{6^n}{5^n} = \left(\frac{6}{5}\right)^n$

$$\rightsquigarrow \sum_{n=1}^{\infty} \frac{6^n}{5^n-1} > \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n \leftsquigarrow \text{geometric series with Common ratio } r=\frac{6}{5}>1$$

$\Rightarrow$ divergent

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$: divergent

(10) $\displaystyle\sum_{K=1}^{\infty} \frac{K\sin^2 K}{1+K^3}$ $\qquad \left.\begin{array}{l}\sin^2 K \leqslant 1 \rightsquigarrow K\sin^2 K \leqslant K \\ 1+K^3 > K^3\end{array}\right\} \Rightarrow \frac{K\sin^2 K}{1+K^3} < \frac{K}{K^3} = \frac{1}{K^2}$

$$\rightsquigarrow \sum_{K=1}^{\infty} \frac{K\sin^2 K}{1+K^3} < \underbrace{\sum_{K=1}^{\infty} \frac{1}{K^2}}_{\text{Convergent}} \Rightarrow \sum_{K=1}^{\infty} \frac{K\sin^2 K}{1+K^3} \text{ : Convergent}$$

(16) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ $\qquad$ for $n \geqslant 2$, $n^n \geqslant n^2 \rightsquigarrow \frac{1}{n^n} \leqslant \frac{1}{n^2}$

$$\rightsquigarrow \sum_{n=1}^{\infty} \frac{1}{n^n} \leqslant \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{Convergent}} \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^n} \text{ Convergent}$$

(18) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$ $\qquad \displaystyle\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\frac{2}{\sqrt{n}+2}} = \lim_{n\to\infty} \frac{\sqrt{n}+2}{2\sqrt{n}} = \frac{1}{2}$

$$\Rightarrow \text{Since } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ is divergent, } \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2} \text{ is divergent.}$$

(40)(a) $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \rightsquigarrow$ There exist a positive number N, such that for any

$n \geqslant N$, $\frac{a_n}{b_n} < 1 \rightsquigarrow a_n < b_n$

$$\rightsquigarrow \sum_{N}^{\infty} a_n < \underbrace{\sum_{N}^{\infty} b_n}_{\text{Convergent}} \rightsquigarrow \sum_{N}^{\infty} a_n \text{ Convergent} \rightsquigarrow \sum a_n \text{ Convergent}$$

(b) (i) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$, $\lim_{n\to\infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ Convergent $\rightsquigarrow$ $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ Convergent

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$ $\lim_{n\to\infty} \frac{\frac{\ln n}{\sqrt{n}\, e^n}}{\frac{1}{e^n}} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = 0$

$\sum_{n=1}^{\infty} \frac{1}{e^n}$: geometric series with $r = \frac{1}{e} < 1$ $\rightsquigarrow$ Convergent

$\rightsquigarrow$ $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$ Convergent.

(41) (a) $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \Rightarrow$ There exist a $N > 0$ Such that for any $n \geq N$

$\frac{a_n}{b_n} \geq 1$

$\Rightarrow \sum_{n=N}^{\infty} a_n \geq \underbrace{\sum_{n=N}^{\infty} b_n}_{\text{divergent}} \rightsquigarrow \sum_{n=N}^{\infty} a_n$ divergent

$\rightsquigarrow \sum_{n=1}^{\infty} a_n$ divergent

(b)(i) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ $\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \infty$

$\sum_{n=2}^{\infty} \frac{1}{n}$ divergent $\Rightarrow$ $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ divergent

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $\lim_{n\to\infty} \frac{\frac{\ln n}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \ln n = \infty$

$\sum_{n=1}^{\infty} \frac{1}{n}$ divergent $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ divergent

(42) Set $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n}$ $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to+\infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = 0$

$\sum \frac{1}{n^2}$ Convergent $\qquad \sum \frac{1}{n}$ divergent

$\boxed{11.5}$

(16) $$\sum_{n=1}^{\infty} \frac{n\cos n\pi}{2^n} = \sum_{n=1}^{\infty} \frac{n(-1)^n}{2^n}$$

$$f(x) = \frac{x}{2^x} \longrightarrow f'(x) = \frac{2^x - x2^x \ln 2}{2^{2x}} = \frac{1 - x\ln 2}{2^x}$$

for $x \geqslant 2$, $f'(x) < 0 \rightsquigarrow \frac{n}{2^n}$ decreasing , $\lim\limits_{n\to+\infty} \frac{n}{2^n} = \lim\limits_{n\to+\infty} \frac{1}{\ln 2 . n . 2^n} = 0$

$\rightsquigarrow$ alternating series test implies $\sum\limits_{n=1}^{\infty} \frac{n(-1)^n}{2^n}$: Convergent .

(20) $$\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n}) = \sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n}) \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+1} + \sqrt{n}} \, .$$

$\sqrt{n+2} > \sqrt{n} \rightsquigarrow \frac{1}{\sqrt{n+1} + \sqrt{n+2}} < \frac{1}{\sqrt{n} + \sqrt{n+1}} \implies \sum\limits_{n=1}^{\infty} (-1)^n . \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$\lim\limits_{n\to+\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}} = 0$ $\qquad$ Convergent

(36) (a)

$$S_{2n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$h_{2n} = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$$

$$\rightsquigarrow S_{2n} - h_{2n} = -2 - \frac{2}{4} - \frac{2}{6} - \cdots - \frac{2}{2n} = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = -h_n$$

$$\rightsquigarrow S_{2n} = h_{2n} - h_n$$

(b) $s_{2n} = h_{2n} - h_n \rightsquigarrow \lim\limits_{n \to \infty} s_{2n} = \lim\limits_{n \to \infty} h_{2n} - h_n = \lim\limits_{n \to \infty} (h_{2n} - \ln(2n)) - (h_n - \ln n) + \ln 2$

$= \gamma - \gamma + \ln 2 = \ln 2$