Homework 8

$\boxed{11.6}$

(9) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{3^n}{2^n n^3}$ $\displaystyle\lim_{n\to\infty}\left|\frac{(-1)^n \frac{3^{n+1}}{2^{n+1}(n+1)^3}}{(-1)^{n-1}\frac{3^n}{2^n n^3}}\right| = \lim_{n\to\infty}\left|\frac{3}{2}\cdot\frac{n^3}{(n+1)^3}\right|$

$$= \frac{3}{2}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3 = \frac{3}{2} > 1$$

$$\rightsquigarrow \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{2^n n^3} : \text{divergent}$$

(20) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2(n+1)!}{(n+1)!^2}\cdot\frac{(n!)^2}{\{2n!\}^2}$

$$= \lim_{n\to\infty}\frac{(2n+1)(2n+2)}{(n+1)^2} = \lim_{n\to\infty}\frac{(2+\frac{1}{n})(2+\frac{2}{n})}{(1+\frac{1}{n})^2} = 4 > 1$$

$$\rightsquigarrow \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2} : \text{divergent}$$

(38) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ $(n+1)\ln(n+1) > n\ln n \longrightarrow \dfrac{1}{(n+1)\ln(n+1)} < \dfrac{1}{n\ln n}$

$$\lim_{n\to\infty}\frac{1}{n\ln n} = 0$$

$$\longrightarrow \text{alternating test} : \sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n} : \text{Convergent}$$

$\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$, $\dfrac{1}{n\ln n}$ decreasing $\underset{\text{integral test}}{\rightsquigarrow}$ $\displaystyle\int_2^{\infty}\frac{1}{x\ln x}\,dx \underset{\substack{u=\ln x\\ du=\frac{1}{x}dx}}{=} \int_{\ln 2}^{\infty}\frac{1}{u}\,du$

$$= \lim_{t\to\infty}\ln t - \ln\ln 2 = \infty$$

$$\rightsquigarrow \sum_{n=2}^{\infty}\frac{1}{n\ln n} \text{ divergent} \rightsquigarrow \sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n} : \text{Conditionally Convergent}$$

(42) $$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 \cdots b_n} \xrightarrow[\substack{\text{Series}\\ \text{of absolute}\\ \text{values}}]{} \sum_{n=1}^{\infty} \frac{n!}{n^n b_1 b_2 \cdots b_n}$$

$$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1} b_1 b_2 \cdots b_n b_{n+1}} \cdot \frac{n^n b_1 b_2 \cdots b_n}{n!} = \lim_{n\to\infty} \frac{(n+1)\cancel{!} \cdot n^n}{(n+1)^{n+1} b_{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{b_{n+1}} = \lim_{n\to\infty} 2 \cdot \left(\frac{n}{n+1}\right)^n = 2 \cdot \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \frac{2}{e} < 1 \rightsquigarrow \text{absolutely Convergent}$$

(45) (a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$

$\rightsquigarrow \sum_{n=0}^{\infty} \frac{x^n}{n!}$ Convergent for all x

(b) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$: Convergent $\rightsquigarrow \lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x

$\boxed{11.8}$ (8) $\sum_{n=1}^{\infty} n^n x^n$ $\quad \underbrace{\lim_{n\to\infty} \sqrt[n]{|n^n x^n|}}_{\text{root test}} = \lim_{n\to\infty} n|x| = \begin{cases} 0 & x=0 \\ \infty & x \neq 0 \end{cases}$

$\rightsquigarrow R = 0 \qquad I = \{0\}$ ← Interval of Convergence

(30) $\sum_{n=0}^{\infty} c_n x^n \xrightarrow[\text{Theorem 4}]{}$ For a $R > 0$ $\begin{cases} |x| < R & \text{Convergent series} \\ |x| > R & \text{divergent series} \end{cases}$

$\rightsquigarrow$ Since the series is Convergent for $x=-4$ and divergent for $x=6$ we have $4 \leq R \leq 6 \rightsquigarrow$ Series is Convergent for $x=1$

$\rightsquigarrow$ (a) $\sum_{n=0}^{\infty} c_n$ Convergent

(b) $4 \leq R \leq 6 \underset{8>6\geq R}{\rightsquigarrow} \sum_{n=0}^{\infty} c_n 8^n$ divergent

(c) $\sum_{n=0}^{\infty} c_n (-3)^n$, $-4 < -3 < 4 \rightsquigarrow$ Convergent

(d) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n = \sum_{n=0}^{\infty} c_n (-9)^n \underset{-9<-6<-R}{\rightsquigarrow}$ divergent

(32) First, we are looking for power series such that after using the ratio test we get the series is Convergent for $\left|x-\frac{p+q}{2}\right| < \frac{q-p}{2}$

$\underbrace{\hspace{4em}}$ Interval of radius $\frac{q-p}{2}$ around $\frac{p+q}{2}$

Therefore, $\sum_{n=1}^{\infty} C_n\left(x-\frac{p+q}{2}\right)^n$ and $\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \frac{2}{q-p}$

(a) Set $C_n = \left(\frac{2}{q-p}\right)^n$ to get a geometric power series:

$$\sum_{n=1}^{\infty}\left(\frac{2}{q-p}\right)^n\left(x-\frac{p+q}{2}\right)^n$$

$\rightsquigarrow$ Convergent for $\frac{2}{q-p}\left|x-\frac{p+q}{2}\right| < 1 \rightsquigarrow \left|x-\frac{p+q}{2}\right| < \frac{q-p}{2}$

$\rightsquigarrow$ Interval of Convergence is (p,q)

(b) Set $x=q \rightsquigarrow \sum_{n=1}^{\infty} C_n\left(\frac{q-p}{2}\right)^n$

$x=p \rightsquigarrow \sum_{n=1}^{\infty} C_n\left(\frac{p-q}{2}\right)^n = \sum_{n=1} C_n(-1)^n\left(\frac{q-p}{2}\right)^n$

$\rightsquigarrow$ Set $C_n = (-1)^n\left(\frac{2}{q-p}\right)^n\cdot\frac{1}{n} \rightsquigarrow$ for $x=q$ we get $\overset{\text{Convergent}}{\text{alternating harmonic Series}}$

for $x=p$ " " $\underline{\text{harmonic Series}}$

$\rightsquigarrow$ divergent

(c) Set $C_n = \left(\frac{2}{q-p}\right)^n\frac{1}{n} \rightsquigarrow$ for $x=p$: alter. harmonic Series $\rightsquigarrow$ Convergent

for $x=q$: harmonic series $\rightsquigarrow$ divergent

(d) Set $C_n = \left(\frac{2}{q-p}\right)^n\cdot\frac{1}{n^2} \rightsquigarrow$ for $x=p$: $\sum_{n=1}^{\infty}\left(\frac{2}{q-p}\right)^n\cdot\frac{1}{n^2}\cdot\left(\frac{q-p}{2}\right)^n = \sum_{n=1}^{\infty}\frac{1}{n^2}$

for $x=q$: $\sum_{n=1}^{\infty}\left(\frac{2}{q-p}\right)^n(-1)^n\left(\frac{q-p}{2}\right)^n\cdot\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

$\rightsquigarrow$ Series Convergent at $[p,q]$

(38) $f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad c_{n+4} = c_n$ for all $n \geqslant 0$

$$\Rightarrow f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_0 x^4 + c_1 x^5 + c_2 x^6 + c_3 x^7 + c_0 x^8 + \cdots$$
$$= \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3\right)\left(\underbrace{1 + x^4 + x^8 + x^{12} + \cdots}_{\text{Convergent} \longleftrightarrow f(x) \text{ Convergent}}\right)$$

$$\sum_{n=0}^{\infty} x^{4n} = 1 + x^4 + x^8 + x^{12} + \cdots \quad \leftsquigarrow \text{ geometric series } \; r = x^4$$

$\rightsquigarrow$ Convergent for $|x^4| < 1 \rightsquigarrow |x| < 1$

at $x = +1$ or $x = -1$, $\sum_{n=0}^{\infty} x^{4n}$ is divergent $\rightsquigarrow \sum_{n=0} c_n x^n$ divergent

$\rightsquigarrow$ Interval of Convergence : $|x| < 1$ i.e. $I = (-1, 1)$

(40) If $\lim_{n \to \infty} \left|\frac{c_n}{c_{n+1}}\right| = L \xrightarrow[\text{ratio test}]{} \lim_{n \to \infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n}\right| =$

$$\lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right| \cdot |x-a| = \frac{|x-a|}{L} \xrightarrow[\text{ratio test implies}]{} \text{Series is Convergent}$$

for $\frac{|x-a|}{L} < 1$ and divergent for $\frac{|x-a|}{L} > 1$

$\rightsquigarrow$ Convergent for $|x-a| < L$ and divergent for $|x-a| > L$

$\rightsquigarrow$ L = radius of Convergence