Homework 9

Section 11.9

(12) $f(x) = \dfrac{2x+3}{x^2+3x+2}$ $\qquad x^2+3x+2 = (x+1)(x+2)$

$\rightsquigarrow f(x) = \dfrac{A}{x+1} + \dfrac{B}{x+2} \rightsquigarrow A(x+2) + B(x+1) = 2x+3$

$\rightsquigarrow \begin{cases} A+B=2 \\ 2A+B=3 \end{cases} \rightsquigarrow A=1,\ B=1$

Therefore $\dfrac{2x+3}{x^2+3x+2} = \dfrac{1}{x+1} + \dfrac{1}{x+2}$

$\dfrac{1}{x+1} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \leftsquigarrow$ Conv. for $|x|<1$

$\dfrac{1}{x+2} = \dfrac{1}{2(1+\frac{x}{2})} = \dfrac{1}{2} \cdot \dfrac{1}{1-(-\frac{x}{2})} = \dfrac{1}{2} \sum_{n=0}^{\infty} (-\frac{x}{2})^n = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{2^{n+1}} x^n$

Convergent for $|-\frac{x}{2}|<1$

$\rightsquigarrow |x|<2$

$\rightsquigarrow \dfrac{2x+3}{x^2+3x+2} = \sum_{n=0}^{\infty} (-1)^n \left(1+\dfrac{1}{2^{n+1}}\right) x^n \leftsquigarrow$ Convergent for $|x|<1$

$\Rightarrow$ Interval of Convergence $(-1,1)$.

(20) $f(x) = \dfrac{x^2+x}{(1-x)^3}$

$g(x) = \dfrac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n$

$g'(x) = \dfrac{1}{(1-x)^2} = 1+2x+3x^2+\cdots = \sum_{n=1}^{\infty} n x^{n-1}$

$g''(x) = \dfrac{2}{(1-x)^3} = 2+6x+12x^2+\cdots = \sum_{n=2}^{\infty} n(n-1) x^{n-2} \leftsquigarrow R=1$

$$\rightsquigarrow f(x) = (x^2+x)\cdot \frac{g''(x)}{2}$$

$$= (x^2+x)\cdot \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^n + \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-1}$$

$$= \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^n + \sum_{n=1}^{\infty} \frac{(n+1)n}{2} x^n = x + \sum_{n=2}^{\infty} n^2 x^n$$

$$= \sum_{n=1}^{\infty} n^2 x^n \rightsquigarrow R=1$$

(28) $\int \frac{\tan^{-1}x}{x}\,dx$

$$f(x) = \tan^{-1}x \rightsquigarrow f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

Convergent for $|x^2|<1$

$$= \sum_{n=0}^{\infty} (-1)^n x^{+2n}$$

$\rightsquigarrow R=1$

$$\rightsquigarrow \tan^{-1}(x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\rightsquigarrow \tan^{-1}(0) = C = 0 \rightsquigarrow \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \rightsquigarrow R=1$$

$$\Rightarrow \frac{\tan^{-1}x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n}$$

$$\rightsquigarrow \int \frac{\tan^{-1}x}{x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} x^{2n+1} \qquad R=1$$

(34)

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \rightsquigarrow f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n\cdot(2n)\, x^{2n-1}}{(2n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} x^{2n-1}$$

$$\rightsquigarrow f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)}{(2n-1)!} x^{2n-2} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n-2)!}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \, x^{2n-2}}{(2n-2)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \, x^{2n}}{(2n)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\rightsquigarrow f''(x) = -f(x) \rightsquigarrow f''(x) + f(x) = 0$$

(40) (a) $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$

$\rightsquigarrow \displaystyle\sum_{n=1}^{\infty} n\, x^{n-1} = \frac{1}{(1-x)^2}$ for $|x|<1$

(b) (i) $\displaystyle\sum_{n=1}^{\infty} n x^n = x \cdot \sum_{n=1}^{\infty} n x^{n-1} = \frac{x}{(1-x)^2}$ for $|x|<1$

(ii) $\overset{x=\frac{1}{2}}{\rightsquigarrow} \displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1/2)^2} = 2$

(c) (i) $\displaystyle\sum_{n=2}^{\infty} n(n-1)\, x^{n-2} = \frac{2}{(1-x)^3}$ for $|x|<1$

$\rightsquigarrow \displaystyle\sum_{n=2}^{\infty} n(n-1)\, x^n = \frac{2x^2}{(1-x)^3}$ $|x|<1$

(ii) $x = \frac{1}{2} \rightsquigarrow \displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} = \frac{2(1/2)^2}{(1/2)^3} = 4$

(iii) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} + \underbrace{\sum_{n=2}^{\infty} \frac{n}{2^n}}_{\left(\sum_{n=1}^{\infty} \frac{n}{2^n}\right) - \frac{1}{2}} = 4 + 2 - \frac{1}{2} = \frac{11}{2}$

$\rightsquigarrow \displaystyle\sum_{n=2}^{\infty} \frac{n^2}{2^n} = \frac{11}{2} \rightsquigarrow \sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{11}{2} + \frac{1}{2} = 6$