Solutions-Problem set 3(section 2.4)

Sunday, February 14, 2016 11:10 AM

$\boxed{10}$

$$\left[\begin{array}{ccc|ccc} 0&0&1&1&0&0\\ 0&1&0&0&1&0\\ 1&0&0&\multicolumn{3}{c}{\underbrace{0\quad 0\quad 1}_{I_3}} \end{array}\right] \xrightarrow[\text{1st \& 3rd rows}]{\text{Switch}} \left[\begin{array}{ccc|ccc} 1&0&0&0&0&1\\ 0&1&0&0&1&0\\ \multicolumn{3}{c|}{\underbrace{0\quad 0\quad 1}_{I_3}}&1&0&0 \end{array}\right] \rightsquigarrow \begin{bmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{bmatrix} \text{ is invertible}$$

and its inverse is equal to $\begin{bmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{bmatrix}$

$\boxed{14}$

$$\left[\begin{array}{cccc|cccc} 1&1&2&3&1&0&0&0\\ 0&-1&0&0&0&1&0&0\\ 2&2&5&4&0&0&1&0\\ 0&3&0&1&0&0&0&1 \end{array}\right] \xrightarrow[\textcircled{3}-2\textcircled{1}]{} \left[\begin{array}{cccc|cccc} 1&1&2&3&1&0&0&0\\ 0&-1&0&0&0&1&0&0\\ 0&0&1&-2&-2&0&1&0\\ 0&3&0&1&0&0&0&1 \end{array}\right] \xrightarrow[\textcircled{1}+\textcircled{2}]{\textcircled{4}+3\textcircled{2}}$$

$$\left[\begin{array}{cccc|cccc} 1&0&2&3&1&1&0&0\\ 0&-1&0&0&0&1&0&0\\ 0&0&1&-2&-2&0&1&0\\ 0&0&0&1&0&3&0&1 \end{array}\right] \xrightarrow[\textcircled{1}-2\textcircled{3}]{-\textcircled{2}} \left[\begin{array}{cccc|cccc} 1&0&0&7&5&1&-2&0\\ 0&1&0&0&0&-1&0&0\\ 0&0&1&-2&-2&0&1&0\\ 0&0&0&1&0&3&0&1 \end{array}\right] \xrightarrow[\textcircled{3}+2\textcircled{4}]{\textcircled{1}-7\textcircled{4}}$$

$$\left[\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\begin{matrix}1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{matrix}} & \begin{matrix}5\\0\\-2\\0\end{matrix} & \begin{matrix}-20\\-1\\6\\3\end{matrix} & \begin{matrix}-2\\0\\1\\0\end{matrix} & \begin{matrix}-7\\0\\2\\1\end{matrix} \end{array}\right] \longrightarrow \text{invertible \& inverse} = \begin{bmatrix} 5&-20&-2&-7\\ 0&-1&0&0\\ -2&6&1&2\\ 0&3&0&1 \end{bmatrix}$$

(the left block above is I_4)

$\boxed{28}$

$$T\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix} = x_1\begin{bmatrix} 22\\ -16\\ 8\\ -5 \end{bmatrix} + x_2\begin{bmatrix} 13\\ -3\\ 9\\ 4 \end{bmatrix} + x_3\begin{bmatrix} 8\\ -2\\ 7\\ 3 \end{bmatrix} + x_4\begin{bmatrix} 3\\ -2\\ 2\\ 1 \end{bmatrix} = \begin{bmatrix} 22&13&8&3\\ -16&-3&-2&-2\\ 8&9&7&2\\ -5&4&3&1 \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 22&13&8&3&1&0&0&0\\ -16&-3&-2&-2&0&1&0&0\\ 8&9&7&2&0&0&1&0\\ -5&4&3&1&0&0&0&1 \end{array}\right] \xrightarrow[\substack{\textcircled{2}+2\textcircled{4}\\ \textcircled{3}-2\textcircled{4}}]{\textcircled{1}-3\textcircled{4}} \left[\begin{array}{cccc|cccc} 37&1&-1&0&1&0&0&-3\\ -26&5&4&0&0&1&0&2\\ 18&1&1&0&0&0&1&-2\\ -5&4&3&1&0&0&0&1 \end{array}\right] \xrightarrow[\textcircled{4}-3\textcircled{3}]{\substack{\textcircled{1}+\textcircled{3}\\ \textcircled{2}-4\textcircled{3}}}$$

$$\left[\begin{array}{cccc|cccc} 55&2&0&0&1&0&1&-5\\ -98&1&0&0&0&1&-4&10\\ 18&1&1&0&0&0&1&-2\\ -59&1&0&1&0&0&-3&7 \end{array}\right] \xrightarrow[\substack{\textcircled{3}-\textcircled{2}\\ \textcircled{4}-\textcircled{2}}]{\textcircled{1}-2\textcircled{2}} \left[\begin{array}{cccc|cccc} 251&0&0&0&1&-2&9&-25\\ -98&1&0&0&0&1&-4&10\\ 116&0&1&0&0&-1&5&-12\\ 39&0&0&1&0&-1&1&-3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{251}\textcircled{1}} \left[\begin{array}{cccc|cccc} 1&0&0&0&\frac{1}{251}&-\frac{2}{251}&\frac{9}{251}&-\frac{25}{251}\\ -98&1&0&0&0&1&-4&10\\ 116&0&1&0&0&-1&5&-12\\ 39&0&0&1&0&-1&1&-3 \end{array}\right] \xrightarrow[\substack{-116\textcircled{1}+\textcircled{3}\\ -39\textcircled{1}+\textcircled{4}}]{98\textcircled{1}+\textcircled{2}} \left[\begin{array}{cccc|cccc} 1&0&0&0&\frac{1}{251}&-\frac{2}{251}&\frac{9}{251}&-\frac{25}{251}\\ 0&1&0&0&\frac{98}{251}&\frac{55}{251}&-\frac{122}{251}&\frac{60}{251}\\ 0&0&1&0&-\frac{116}{251}&-\frac{19}{251}&\frac{211}{251}&-\frac{112}{251}\\ 0&0&0&1&-\frac{39}{251}&-\frac{173}{251}&-\frac{100}{251}&\frac{222}{251} \end{array}\right]$$

(the right block of the last matrix: inverse)

$\boxed{38}$ $\boxed{a}$ A is invertible iff $\mathrm{rref}(A) = I_3$. Thus $a \neq 0$ (if $a = 0$, all entries in 1st Column are 0, therefore the 1st Column of $\mathrm{rref}(A) = 0$! $\underset{o}{X}$)

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & b/a & c/a \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

If $d=0 \rightsquigarrow \begin{bmatrix} 1 & b/a & c/a \\ 0 & 0 & e \\ 0 & 0 & f \end{bmatrix} \rightsquigarrow$ 2^{nd} Column of $\mathrm{rref}(A)$ Can't have a leading 1! X.

Thus $d \neq 0$ $\begin{bmatrix} 1 & b/a & c/a \\ 0 & 1 & e/d \\ 0 & 0 & f \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & c/a - be/ad \\ 0 & 1 & e/d \\ 0 & 0 & f \end{bmatrix}$ with the same reason $f \neq 0$

(if $f=0$ then 3^{rd} Column of $\mathrm{rref}(A)$ doesn't have a leading 1! X.)

$\Rightarrow f \neq 0 \longrightarrow \begin{bmatrix} 1 & 0 & c/a - be/ad \\ 0 & 1 & e/d \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

<u>Therefore</u>: A is invertible iff $a, d, f \neq 0$

$\boxed{b}$ An upper triangular matrix A (of Size $n \times n$) is invertible iff the diagonal entries are nonzero (i.e. for any $1 \leq i \leq n$ $A_{ii} \neq 0$)

$\boxed{c}$ <u>Yes</u> Consider $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ 0 & \cdots & & a_{nn} \end{bmatrix}$ and assume $a_{11}, \dots, a_{nn}$ are <u>nonzero</u>.

$$[A \mid I_n] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right] \xrightarrow[\frac{1}{a_{nn}} ⓝ]{\frac{1}{a_{11}} ① \; \frac{1}{a_{22}} ② \; \vdots}$$

$$\left[\begin{array}{cccc|cccc} 1 & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} & 1/a_{11} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & a_{2n}/a_{22} & 0 & 1/a_{22} & \cdots & 0 \\ \vdots & & & & \vdots & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1/a_{nn} \end{array}\right]$$

in order to compute the reduced row echelon form, we should step by step (for $i = 2, \dots, n$) subtract appropriate coefficient of the ith row from the $1, \dots, i-1$ row to make entries $a_{1i}/a_{11}, \dots, a_{i-1\,i}/a_{i-1\,i-1}$ equal zero. Note that these moves makes the $n \times n$ matrix in the right upper triangular $\Rightarrow A^{-1}$ is upper triangular.

$\boxed{44}$ a.

$$M_4=\begin{bmatrix}1&5&9&13\\2&6&10&14\\3&7&11&15\\4&8&12&16\end{bmatrix}\xrightarrow[\substack{\textcircled{3}-3\textcircled{1}\\\textcircled{4}-4\textcircled{1}}]{\textcircled{2}-2\textcircled{1}}\begin{bmatrix}1&5&9&13\\0&-4&-8&-12\\0&-8&-16&-24\\0&-12&-24&-36\end{bmatrix}\xrightarrow[\substack{-\frac{1}{8}\textcircled{3}\\-\frac{1}{12}\textcircled{4}}]{-\frac{1}{4}\textcircled{2}}\begin{bmatrix}1&5&9&13\\0&1&2&3\\0&1&2&3\\0&1&2&3\end{bmatrix}$$

$$\xrightarrow[\substack{\textcircled{3}-\textcircled{2}\\\textcircled{4}-\textcircled{2}}]{\textcircled{1}-5\textcircled{2}}\begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{bmatrix}\longrightarrow \operatorname{rank}(M_4)=2$$

b.

$$M_n=\begin{bmatrix}1&n+1&2n+1&\cdots&(n-1)n+1\\2&n+2&2n+2&\cdots&(n-1)n+2\\\vdots&\vdots&\vdots&&\\i&n+i&2n+i&\cdots&(n-1)n+i\\\vdots&\vdots&\vdots&&\vdots\\n&2n&3n&\cdots&n^2\end{bmatrix}\xrightarrow[\substack{\vdots\\\textcircled{n}-n\textcircled{1}}]{\substack{\textcircled{2}-2\textcircled{1}\\\textcircled{3}-3\textcircled{1}}}\begin{bmatrix}1&n+1&2n+1&\cdots&(n-1)n+1\\0&-n&-2n&\cdots&-(n-1)n\\\vdots&&&&\\0&(1-i)n&(1-i)2n&\cdots&(1-i)(n-1)n\\\vdots&&&&\\0&(1-n)n&(1-n)2n&\cdots&(1-n)(n-1)n\end{bmatrix}$$

$$\xrightarrow[\substack{\frac{1}{(1-i)n}\textcircled{i}\\\vdots\\\frac{1}{(1-n)n}\textcircled{n}}]{\substack{-\frac{1}{n}\textcircled{2}\\\vdots}}\begin{bmatrix}1&n+1&2n+1&\cdots&(n-1)n+1\\0&1&2&\cdots&n-1\\\vdots&&&&\\0&1&2&\cdots&n-1\\\vdots&&&&\\0&1&2&\cdots&n-1\end{bmatrix}\xrightarrow[\substack{\vdots\\\textcircled{n}-\textcircled{2}}]{\substack{\textcircled{1}-(n+1)\textcircled{2}\\\textcircled{3}-\textcircled{2}}}\begin{bmatrix}1&0&-1&\cdots&-n+2\\0&1&2&\cdots&n-1\\0&0&0&\cdots&0\\0&0&0&\cdots&0\\\vdots&&&&\\0&0&0&\cdots&0\end{bmatrix}$$

$\Rightarrow \operatorname{rank}(M_n)=2$

c. M_1 : invertible, for $n\geq 2$, M_n invertible $\longleftrightarrow \underbrace{\operatorname{rank}(M_n)}_{\substack{\shortparallel\\2}}=n$

$\Rightarrow n=2$

$\rightsquigarrow M_n$ is invertible for $n=1,2$.