Solutions-Problem set 3(section 3.1)

Sunday, February 14, 2016 2:24 PM

$\boxed{10}$

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & -2 & 2 & 0 \\ 1 & -1 & -2 & 0 & 3 & 0 \\ 2 & -2 & -1 & 3 & 4 & 0 \end{array}\right] \xrightarrow[\text{④}-2\text{①}]{\text{②}+\text{①},\ \text{③}-\text{①}} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 3 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{-\text{②}} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right] \xrightarrow[\text{④}-\text{②}]{\text{①}+\text{②},\ \text{③}+\text{②}} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{-\text{③}} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{array}\right] \xrightarrow[\text{②}+3\text{③},\ \text{④}-5\text{③}]{\text{①}+2\text{③}} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\longrightarrow x_2, x_4$: free variable $\rightsquigarrow \begin{cases} x_1 = t - 2s \\ x_2 = t \\ x_3 = -s \\ x_4 = s \\ x_5 = 0 \end{cases} \longrightarrow \ker(A) = \left\{ t\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix} + s\begin{bmatrix}-2\\0\\-1\\1\\0\end{bmatrix} \;\middle|\; t, s \in \mathbb{R} \right\}$

$$= \operatorname{Span}\left(\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\\1\\0\end{bmatrix} \right)$$

$\boxed{15}$ $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \longrightarrow \operatorname{im}(A) = \operatorname{Span}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\4\end{bmatrix} \right)$

$\begin{bmatrix}1\\1\end{bmatrix} \neq 0$ ✓ $\begin{bmatrix}1\\2\end{bmatrix}$: not a multi. of $\begin{bmatrix}1\\1\end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is $\underset{(\det = 1)}{\underline{\text{invertible}}}$, therefore for any vector $\begin{bmatrix}a\\b\end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$ has a solution.

Thus, both $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}1\\4\end{bmatrix}$ are a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix} \Rightarrow \begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}1\\4\end{bmatrix}$ are redundant. $\rightsquigarrow \operatorname{im}(A) = \operatorname{Span}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right) = \operatorname{im}\Big(\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{\text{invertible}} \Big) = \mathbb{R}^2$

$\boxed{19}$ $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$ $\quad \operatorname{im}(A) = \operatorname{Span}\left(\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}, \begin{bmatrix}3\\-6\end{bmatrix}, \begin{bmatrix}4\\-8\end{bmatrix} \right)$

$\begin{bmatrix}2\\-4\end{bmatrix} = 2\begin{bmatrix}1\\-2\end{bmatrix}$, $\begin{bmatrix}3\\-6\end{bmatrix} = 3\begin{bmatrix}1\\-2\end{bmatrix}$ $\begin{bmatrix}4\\-8\end{bmatrix} = 4\begin{bmatrix}1\\-2\end{bmatrix} \Rightarrow \operatorname{im}(A) = \operatorname{Span}\left(\begin{bmatrix}1\\-2\end{bmatrix} \right) = \left\{ t\begin{bmatrix}1\\-2\end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$: $\underline{\underline{\text{line}}}$.

34 $\begin{bmatrix}-1\\1\\2\end{bmatrix}$: satisfies these equations : $\begin{cases} x_1+x_2=0\\ 2x_1+x_3=0\end{cases} \rightarrow \begin{bmatrix}1&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$

Therefore, $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}1&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

(Note that this a homo. System and all of the scalar multi. of $\begin{bmatrix}-1\\1\\2\end{bmatrix}$ i.e. line spanned by $\begin{bmatrix}-1\\1\\2\end{bmatrix}$ satisfy these equations. Infact, line spanned by $\begin{bmatrix}-1\\1\\2\end{bmatrix}$ is the set of solutions of the system)

42 Let's use the hint : $\operatorname{im}(A)=\{\vec{y}\in\mathbb{R}^4 \mid \text{the system is Consistent}\}$

$$=\left\{\vec{y}\in\mathbb{R}^4 \;\middle|\; \begin{cases} y_1 \quad -3y_3+2y_4=0\\ y_2-2y_3+y_4=0\end{cases}\right\}$$

$$=\left\{\vec{y}=\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix} \;\middle|\; \begin{bmatrix}1&0&-3&2\\0&1&-2&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\right\}$$

$$=\operatorname{Ker}\left(\begin{bmatrix}1&0&-3&2\\0&1&-2&1\end{bmatrix}\right)$$

44 A , $B=\operatorname{rref}(A)$

a Yes, $\operatorname{Ker}(A)=\{\vec{x}\mid A\vec{x}=\vec{0}\}$

$\operatorname{Ker}(B)=\{\vec{x}\mid B\vec{x}=\vec{0}\}$ & $\operatorname{rref}([A\,\vdots\,\vec{0}])=[\operatorname{rref}(A)\,\vdots\,\vec{0}]=[B\,\vdots\,\vec{0}]$

Elementary row operations doesn't change the solutions of a linear system. After applying them to $[A\,\vdots\,\vec{0}]$ to Compute it's rref, we get $[B\,\vdots\,\vec{0}]$. Therefore $\operatorname{Ker}(A)=\operatorname{Ker}(B)$.

b No, for example $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ $\operatorname{im}(A)=\operatorname{Span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$

$\neq$

$B=\operatorname{rref}(A)=\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $\operatorname{im}(B)=\operatorname{Span}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$

48 **a** $\vec{w}$ is in $\operatorname{im}(A)$, thus $\vec{w}=A\vec{v}$ for a vector $\vec{v}$. $\rightsquigarrow A\vec{w}=A(A\vec{v})=A^2\vec{v}=A\vec{v}=\vec{w}$

$\Rightarrow \vec{w}=A\vec{w}$

b if $\operatorname{rank}(A)=2 \Rightarrow A$ is invertible, multiply $A^2=A$ by $A^{-1} \Rightarrow \boxed{A=I_2}$

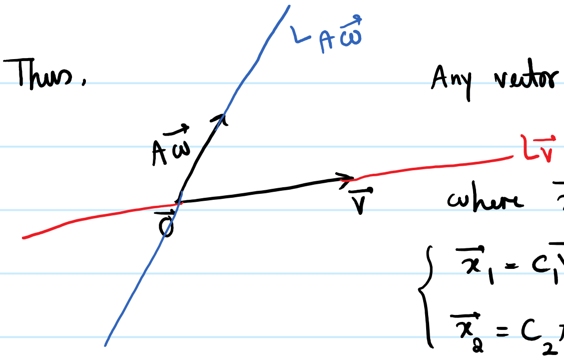
if $\operatorname{rank}(A)=0 \Rightarrow \operatorname{rref}(A)=\begin{bmatrix}0&0\\0&0\end{bmatrix} \Rightarrow A=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

c $\operatorname{rank}(A)=1 \Rightarrow \operatorname{rref}(A)=\begin{bmatrix}1&a\\0&0\end{bmatrix}$ or $\operatorname{rref}(A)=\begin{bmatrix}0&1\\0&0\end{bmatrix} \rightsquigarrow$ in both Cases, there exist one vector $\vec{v}$ such that $\operatorname{Ker}(A)=\operatorname{Span}(\vec{v})$.

pick $\vec{w}$ in $\mathbb{R}^2$ s.t. $\vec{w}$ is not on the line spanned by $\vec{v}$, thus $A\vec{w}\neq\begin{bmatrix}0\\0\end{bmatrix}$ and $A^2\vec{w}=A\vec{w}$

$\Rightarrow A(A\vec{w})=A\vec{w} \Rightarrow A\vec{w}\notin\operatorname{Ker}(A)$

Thus,



Any vector $\vec{x}$ in $\mathbb{R}^2$ can be expressed uniquely as

$$\vec{x} = \vec{x}_1 + \vec{x}_2$$

where $\vec{x}_1$ is on $L_{\vec{v}}$ and $\vec{x}_2$ is on $L_{A\vec{\omega}}$.

$$\begin{cases} \vec{x}_1 = c_1\vec{v} \\ \vec{x}_2 = c_2 A\vec{\omega} \end{cases} \longrightarrow \begin{array}{l} A\vec{x}_1 = A c_1\vec{v} = c_1 A\vec{v} = \vec{0} \\ A\vec{x}_2 = A c_2 A\vec{\omega} = c_2 A^2\vec{\omega} = c_2 A\vec{\omega} \end{array}$$

$$\Rightarrow A\vec{x} = \vec{0} + c_2 A\vec{\omega} = \vec{x}_2 \quad \square$$