Solutions-Problem set 4(section 3.3)

Wednesday, March 2, 2016 10:26 AM

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$$\begin{bmatrix} 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} \xrightarrow[\substack{(2)-3(4) \\ (3)-2(4)}]{(1)-4(4)} \begin{bmatrix} 0 & 0 & -11 & -7 & 6 \\ 0 & 0 & -8 & -4 & 5 \\ 0 & 0 & -5 & 5 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} \xrightarrow[-\frac{1}{5}(3)]{} \begin{bmatrix} 0 & 0 & -11 & -7 & 6 \\ 0 & 0 & -8 & -4 & 5 \\ 0 & 0 & 1 & -1 & -2 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{(2)+8(3) \\ (4)-3(3)}]{(1)+11(3)} \begin{bmatrix} 0 & 0 & 0 & -18 & -16 \\ 0 & 0 & 0 & -12 & -11 \\ 0 & 0 & 1 & -1 & -2 \\ 1 & 2 & 0 & 5 & 6 \end{bmatrix} \xrightarrow{-\frac{1}{18}(1)} \begin{bmatrix} 0 & 0 & 0 & 1 & \frac{8}{9} \\ 0 & 0 & 0 & -12 & -11 \\ 0 & 0 & 1 & -1 & -2 \\ 1 & 2 & 0 & 5 & 6 \end{bmatrix} \xrightarrow[\substack{(3)+(1) \\ (4)-5(1)}]{(2)+12(1)}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & \frac{8}{9} \\ 0 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & -\frac{10}{9} \\ 1 & 2 & 0 & 0 & \frac{14}{9} \end{bmatrix} \xrightarrow[-3(2)]{} \begin{bmatrix} 0 & 0 & 0 & 1 & \frac{8}{9} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{10}{9} \\ 1 & 2 & 0 & 0 & \frac{14}{9} \end{bmatrix} \xrightarrow[\substack{(3)+\frac{10}{9}(2) \\ (4)-\frac{14}{9}(2)}]{(1)-\frac{8}{9}(2)}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{rearrange}]{} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccccc} & \underset{\text{redundant}}{\uparrow} & & & \end{array}$$

$$\text{Basis for } \operatorname{im}(A) = \left(\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 10 \\ 0 \end{bmatrix} \right)$$

$$\operatorname{Ker}(A) = \left\{ \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

basis for ker(A)

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$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \vec{v}_2 - \vec{v}_3 = \vec{0} \rightsquigarrow \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in \ker(C)$$

$$\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$

$\vec{v}_2 - \vec{v}_3 = \vec{0}$ is the only relation btn $\vec{v}_1, \vec{v}_2, \vec{v}_3 \Rightarrow \ker(C) = \operatorname{Span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$

Ker of a 3x3 matrix is spanned by $\begin{bmatrix}0\\1\\-1\end{bmatrix}$ iff the 1st and second Column are linearly indep. and the 2nd and 3rd Column are equal.

$\Rightarrow$ Just L has the same kernel as C.

$$Im(C) = Span\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right) = Span\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right) = Span\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

$\leadsto Im(C) = Im(H) = Im(X)$

$$Im(T) = Span\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right) = Span\left(\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right) = Im(Y)$$

$\Rightarrow$ L is the matrix whose image is different from all the other matrices.

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$$\begin{bmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\2&3&4&K\end{bmatrix} \xrightarrow{④-2①} \begin{bmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\0&3&4&K-4\end{bmatrix} \xrightarrow{④-3②} \begin{bmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\0&0&4&K-13\end{bmatrix} \xrightarrow{④-4③}$$

$$\begin{bmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\0&0&0&K-29\end{bmatrix} \Rightarrow \text{invertible iff } K \neq 29$$

$\Rightarrow$ for any $K \neq 29$, vectors are linearly indep.

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$$2x_1 - x_2 + 2x_3 + 4x_4 = 0 \leadsto x_1 - \frac{x_2}{2} + x_3 + 2x_4 = 0 \Rightarrow \begin{bmatrix}\frac{t}{2} - s - 2r\\t\\s\\r\end{bmatrix} = t\begin{bmatrix}\frac{1}{2}\\1\\0\\0\end{bmatrix} + s\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + r\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}$$

$$\leadsto \text{basis} = \left(\begin{bmatrix}1/2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\1\end{bmatrix}\right)$$

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$$\vec{x} \cdot \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} = 0 \text{ and } \vec{x} \cdot \begin{bmatrix}0\\1\\2\\3\end{bmatrix} = 0 \leadsto \begin{bmatrix}1&0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = 0 \text{ and } \begin{bmatrix}0&1&2&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = 0$$

$$\leadsto \begin{bmatrix}1&0&-1&1\\0&1&2&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = 0 \longrightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}t-r\\-2t-3r\\t\\r\end{bmatrix} = Span\Bigg(\underbrace{\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\-3\\0\\1\end{bmatrix}}_{\text{basis}}\Bigg)$$

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$Im(A) = Span(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q) = V$

After removing redundant vectors of $(\vec{v}_1 \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q)$ we get a basis for V. Since $\vec{v}_1, \dots, \vec{v}_p$ are linearly indep. no redundant vector between $\vec{v}_1, \dots, \vec{v}_p$. Therefore, the obtained basis Consists of all $\vec{v}_1, \dots, \vec{v}_p$.

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$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{③-3① \\ ④-4①}]{②-2①} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 1 & 0 & 0 \\ 0 & 3 & -3 & 0 & 1 & 0 \\ 0 & 4 & -4 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 0 & 0 \\ 0 & 3 & -3 & 0 & 1 & 0 \\ 0 & 4 & -4 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{③-3② \\ ④-4②}]{①-②} \begin{bmatrix} 1 & 0 & 2 & -1/2 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -3/2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -1/2 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{②-\frac{1}{2}③ \\ ④+2③}]{①+\frac{1}{2}③}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1/3 & 0 \\ 0 & 1 & -1 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & -4/3 & 1 \end{bmatrix} \xrightarrow[-\frac{3}{4}④]{} \begin{bmatrix} 1 & 0 & 2 & 0 & -1/3 & 0 \\ 0 & 1 & -1 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3/4 \end{bmatrix} \xrightarrow[③+\frac{2}{3}④]{\substack{①+\frac{1}{3}④ \\ ②-\frac{1}{3}④}}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -1/4 \\ 0 & 1 & -1 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & -3/4 \end{bmatrix} \rightsquigarrow \text{basis} = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

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$\text{rank}(A) = \dim \text{Im}(A)$ $\rightsquigarrow$ $\dim \text{Im}(A) = 2$ $\rightsquigarrow$ $\text{rank}(A) = 2$

A : projection on a plane