Solutions-Problem set 4(section 3.4)

Wednesday, March 2, 2016 4:48 PM

$\boxed{16}$

$$\begin{bmatrix}3\\7\\13\end{bmatrix} = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} \rightsquigarrow \begin{cases} c_1 = 3 \\ c_1 + c_2 = 7 \longrightarrow c_2 = 4 \longrightarrow [\vec{x}]_\beta = \begin{bmatrix}3\\4\\6\end{bmatrix} \\ c_1 + c_2 + c_3 = 13 \longrightarrow c_3 = 6 \end{cases}$$

$\boxed{24}$ ⓐ

$$S = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix} \rightsquigarrow S^{-1} = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix} \Rightarrow B = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}\underbrace{\begin{bmatrix}13 & -20\\6 & -9\end{bmatrix}\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}}_{\begin{bmatrix}6 & 5\\3 & 3\end{bmatrix}} = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}$$

ⓑ

$$[\vec{x}]_\beta = \begin{bmatrix}c_1\\c_2\end{bmatrix} \xrightarrow{B = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}} [A\vec{x}]_\beta = \begin{bmatrix}3c_1\\c_2\end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \uparrow$$

$$c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}2c_1+5c_2\\c_1+3c_2\end{bmatrix} \xrightarrow[\begin{bmatrix}13 & -20\\6 & -9\end{bmatrix}]{} \begin{bmatrix}13 & -20\\6 & -9\end{bmatrix}\begin{bmatrix}2c_1+5c_2\\c_1+3c_2\end{bmatrix} = \begin{bmatrix}6c_1+5c_2\\3c_1+3c_2\end{bmatrix}$$

$$= 3c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}5\\3\end{bmatrix}$$

ⓒ

$$A\vec{v}_1 = \begin{bmatrix}13 & -20\\6 & -9\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}6\\3\end{bmatrix} = 3\vec{v}_1 \rightsquigarrow [A\vec{v}_1]_\beta = \begin{bmatrix}3\\0\end{bmatrix}$$

$$A\vec{v}_2 = \begin{bmatrix}13 & -20\\6 & -9\end{bmatrix}\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix} = \vec{v}_2 \rightsquigarrow [A\vec{v}_2]_\beta = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}$$

$\boxed{30}$

$$A\vec{v}_1 = \begin{bmatrix}0 & 2 & -1\\2 & -1 & 0\\4 & -4 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} = \vec{v}_1 \qquad A\vec{v}_2 = \begin{bmatrix}0 & 2 & -1\\2 & -1 & 0\\4 & -4 & 1\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}0\\-1\\-2\end{bmatrix} = -\vec{v}_2$$

$$A\vec{v}_3 = \begin{bmatrix}0 & 2 & -1\\2 & -1 & 0\\4 & -4 & 1\end{bmatrix}\begin{bmatrix}1\\2\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightsquigarrow B = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}$$

$\boxed{36}$

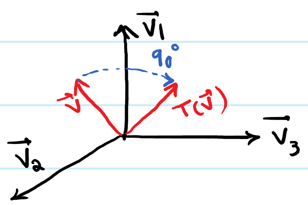
$$T(\vec{x}) = \vec{v}_1 \times \vec{x} + (\vec{v}_1 \cdot \vec{x})\vec{v}_1$$

$$T(\vec{v}_1) = \vec{v}_1 \times \vec{v}_1 + (\vec{v}_1 \cdot \vec{v}_1)\vec{v}_1 = \vec{v}_1$$

$$T(\vec{v}_2) = \vec{v}_1 \times \vec{v}_2 + (\vec{v}_1 \cdot \vec{v}_2)\vec{v}_1 = \vec{v}_3$$

$$T(\vec{v}_3) = \vec{v}_1 \times \vec{v}_3 + (\vec{v}_1 \cdot \vec{v}_3)\vec{v}_1 = -\vec{v}_2$$

$$\rightsquigarrow \beta\text{-matrix}: \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}$$



$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \rightsquigarrow \underbrace{T(\vec{v})}_{\text{rotation of } \vec{v} \text{ about the line spanned by } \vec{v}_1 \text{ by 90 degrees.}} = c_1\vec{v}_1 - c_3\vec{v}_2 + c_2\vec{v}_3$$

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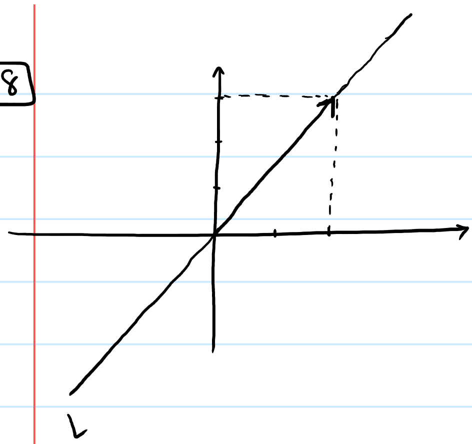

$\beta = (\vec{u}_1, \vec{u}_2)$, if β-matrix of T is diagonal, equal to to $\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ $\rightsquigarrow$ then $[T(\vec{u}_1)]_\beta = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

$[T(\vec{u}_2)]_\beta = \begin{bmatrix} 0 \\ c_2 \end{bmatrix}$

$\Rightarrow T(\vec{u}_1) = c_1 \vec{u}_1$ and $T(\vec{u}_2) = c_2 \vec{u}_2$

we should find linearly indep. vectors $\vec{u}_1$ and $\vec{u}_2$ s.t. $T(\vec{u}_i)$ is parallel to $\vec{u}_i$ for $i = 1, 2$.

$\Rightarrow \vec{u}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $T(\vec{u}_1) = \vec{u}_1$ $\rightsquigarrow$ for $\beta = \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right)$

perp. to $\vec{u}_1$: $\vec{u}_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $T(\vec{u}_2) = -\vec{u}_2$

the β-matrix of T is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

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$\beta = (\vec{v}_1, \vec{v}_2) \rightsquigarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\vec{v}_1 + 5\vec{v}_2$

$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2\vec{v}_1 + 3\vec{v}_2$

$\rightsquigarrow \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = -\begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$

$\rightsquigarrow \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 12 & -7 \\ 14 & -8 \end{bmatrix}$

$\Rightarrow \beta = \left(\begin{bmatrix} 12 \\ 14 \end{bmatrix}, \begin{bmatrix} -7 \\ -8 \end{bmatrix} \right)$

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$T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{x} \rightsquigarrow T(\vec{v}_1) = 5\vec{v}_1$

$T(\vec{v}_2) = -\vec{v}_2$

$\vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{cases} x_1 = t \\ x_2 = 2t \end{cases}$

$\rightsquigarrow$ set $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{cases} x_1 = t \\ x_2 = -t \end{cases}$

$\rightsquigarrow$ set $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow \beta = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$

$\boxed{72}$ $B = S^{-1}AS$

First, $\dim \ker(A) = \dim \ker(B)$ because,

- If $\vec{x}$ in $\ker(B)$, $B\vec{x}=0$, thus $S^{-1}AS\vec{x}=0 \Rightarrow AS\vec{x}=0 \Rightarrow S\vec{x} \in \ker(A)$
- Let $\beta = (\vec{v}_1, \dots, \vec{v}_p)$ be a basis for $\ker(B)$. Since $S\vec{v}_1, \dots, S\vec{v}_p$ are in $\ker(A)$

  $\mathrm{Span}(S\vec{v}_1, \dots, S\vec{v}_p) \subset \ker(A)$

  Since S is invertible, if $c_1 S\vec{v}_1 + \dots + c_p S\vec{v}_p = \vec{0} \leadsto c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}$

  $\longrightarrow c_1 = \dots = c_p = 0$

  Therefore $S\vec{v}_1, \dots, S\vec{v}_p$ are linearly indep.

  If $\vec{y}$ in $\ker(A)$, then $AS(S^{-1}\vec{y}) = \vec{0} \leadsto B(S^{-1}\vec{y}) = \vec{0} \leadsto S^{-1}\vec{y} \in \ker(B)$

  $\Rightarrow S^{-1}\vec{y} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p \Rightarrow \vec{y} = c_1 S\vec{v}_1 + \dots + c_p S\vec{v}_p$

  $\Rightarrow \vec{y} \in \mathrm{Span}(S\vec{v}_1, \dots, S\vec{v}_p)$

  Thus $\ker(A) = \mathrm{Span}(S\vec{v}_1, \dots, S\vec{v}_p) \Rightarrow \dim \ker(A) = p = \dim \ker(B)$

- $\dim \ker(A) + \mathrm{rank}(A) = n$
  $\dim \ker(B) + \mathrm{rank}(B) = n$ $\xrightarrow[\substack{\dim \ker(A) \\ \parallel \\ \dim \ker(B)}]{}$ $\mathrm{rank}(A) = \mathrm{rank}(B)$