Lecture 1

Tuesday, January 17, 2017 5:06 PM

Introduction:

Alg. topology is studying topological spaces via abstract algebra

Top. space $\longmapsto$ Algebraic invariants, invariant up to homomorphism OR homotopy equivalence

* As in the book, maps between spaces are continuous unless stated otherwise.

Def • <u>homotopy</u>: family $f_t : X \longrightarrow Y$ for $t \in [0,1]$ such that $F: X \times [0,1] \longrightarrow Y$ given by $F(x,t) = f_t(x)$ is continuous.

• Maps $f, g : X \longrightarrow Y$ called <u>homotopic</u> if there exist a homotopy $f_t : X \longrightarrow Y$ s.t. $f = f_0$, $g = f_1$, $f \simeq g$

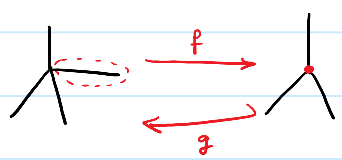
• <u>homotopy equivalence</u>: $f : X \longrightarrow Y$ is a homotopy equivalence if $\exists\, g : Y \to X$ s.t. $fg \simeq \mathbb{I}_{\text{id}\, Y}$ $\quad gf \simeq \mathbb{I}_X$

* Any homeomorphism is a homotopy equivalence.

• X and Y are called homotopy equivalent if $\exists$ a homotopy equivalence $f : X \longrightarrow Y$

Ex (1) for any $n \geq 0$, $\mathbb{R}^n$ is homotopy equivalent with $\{0\}$ (Exercise)

(2)



homotopy equivalent but not homeomorphic.

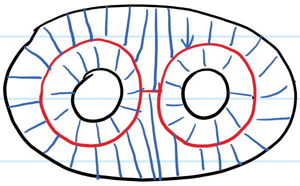
$gf = \text{id}$ $\quad fg \simeq \text{id}$

Special case of homotopy equivalence called <u>deformation retraction</u>:

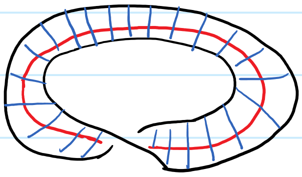
• $A \subset X$ subspace, $r : X \longrightarrow X$ s.t. $r(X) = A$, for any $x \in A$, $r(x) = x$ i.e. $r|_A = \mathbb{I}$

• A deformation retraction of X onto A is a homotopy $f_t : X \longrightarrow X$ between $\mathbb{1}$ and a contraction "r" of X onto A s.t. $f_t|_A = \mathbb{I}$ for any t.
A is called a deformation retract of X.

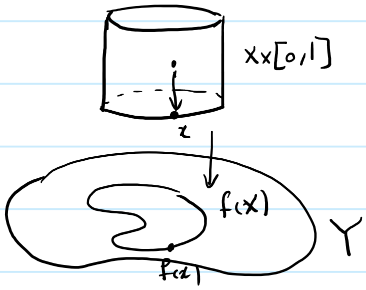
Ex



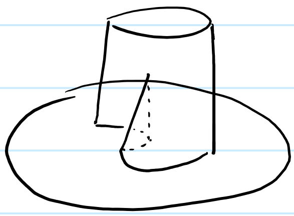
Ex



Ex <u>Mapping cylinder</u> $f: X \longrightarrow Y$ $\quad M_f = X \times [0,1] \sqcup Y / (x,1) \sim f(x)$



$\rightsquigarrow$



$: M_f$

M_f deformation retracts onto Y. by s

- X is called <u>Contractible</u> if $\mathbb{1}: X \longrightarrow X$ is nullhomotopic i.e. $\mathbb{1} \simeq$ Constant map.

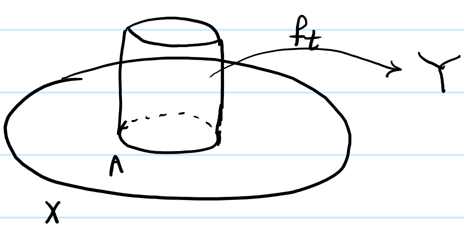
$\Longleftrightarrow$ X has the homotopy type of a pt.

$$X \underset{i}{\overset{f}{\rightleftarrows}} P^{\;e^X} \Rightarrow fi = \mathbb{1}, \ \underset{\mathbb{1}}{\overset{fi}{\simeq}} : X \underset{\text{Constant}}{\longrightarrow} X$$

Ex D_n: Contractible

Q $A \subset X$ Contractible subspace, is $q: X \longrightarrow X/A$ a homotopy equivalence?

<u>Homotopy extension property</u>: (X, A) have homotopy extension property if for any map $f_0: X \longrightarrow Y$ and any homotopy $f_t: A \longrightarrow Y$ of $f_0|_A$, there exists an extension $f_t: X \longrightarrow Y$. for f_0.



<u>Lemma</u> (X, A) has homotopy extension property $\Longleftrightarrow$ $A \times I \cup X \times \{0\}$ is a <u>retract</u> of $X \times I$. "r"

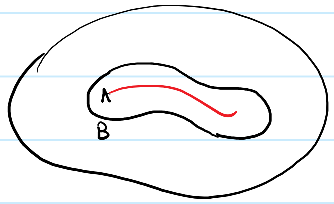
$(\Rightarrow)$ $Y = A \times I \cup X \times \{0\}$ $\quad \mathbb{1}: A \times I \cup X \times \{0\} \longrightarrow A \times I \cup X \times \{0\}$

$\rightsquigarrow$ extends to a map $X \times I \longrightarrow A \times I \cup X \times \{0\}$ retracts!

$(\Leftarrow)$ A Closed Subspace $F: A \times I \cup X \times \{0\} \underset{\text{Conti}}{\longrightarrow} Y \Rightarrow$ For r: extended retraction.

Cor (X,A) has homotopy extension property if A has a mapping cylinder nbd in X.

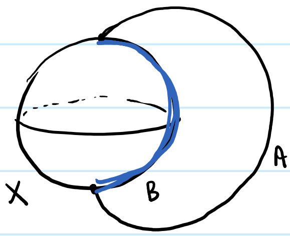
i.e. $\exists$ closed set $N \subset X$ containing A and a subspace B such that $N - B$: open nbd of A and $\exists$ homeo $h : M_f \longrightarrow N$ for $f : B \longrightarrow A$ s.t. $h|_{A \cup B} = \mathbb{1}$



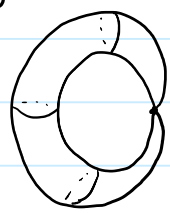
X

Prop : If (X,A) satisfy homotopy extension property and A is contractible the $q : X \longrightarrow X/A$ is a homotopy equivalence.

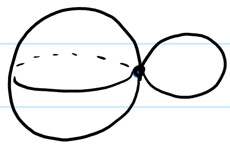
Ex



$\rightsquigarrow X/A$



$\rightsquigarrow X/B$:



$\longleftarrow$ Wedge Sum : $S^2 \vee S^1$

Def : $\underset{x_0}{X}, \underset{y_0}{Y}$, wedge sum of X and Y : $X \amalg Y / x_0 \sim y_0$

Prop $A \subset X \cap Y$ such that (X,A) and (Y,A) have homotopy extension property. $f : X \longrightarrow Y$ homotopy equivalence such that $f|_A = \mathbb{1}$

$\Rightarrow$ f is a homotopy equivalence <u>relative</u> A.

i.e. $\exists g : Y \to X$ s.t. $g|_A = \mathbb{1}$ and $fg \simeq \mathbb{1}$ $gf \simeq \mathbb{1}$ s.t. homotopies are equal to $\mathbb{1}$ on A at all times.

e.g. deformation retraction is a relative homotopy between $\mathbb{1}$ and a retraction map.

Cor $A \subset X$ subspace s.t. $i : A \hookrightarrow X$ is a homotopy equivalence and (X,A) satisfies hom. ext. prop. then A is a deform. retract of X.

Cor $f : X \longrightarrow Y$ homotopy equivalence $\Longleftrightarrow$ X is a deformation retract of M_f.

X has a mapping cylinder nbd in $M_f \Rightarrow (M_f, X)$ homotopy exten. property

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i"'] & Y \arrow[d, shift right, "j"'] \\ & M_f \arrow[u, shift right, "r"'] \end{tikzcd}$$

$f = ri \Rightarrow$ if i is homotopy equivalence $\Rightarrow$ f homotopy equi.

$i \underset{(EX)}{\simeq} jf \Rightarrow$ if f homotopy equivalence $\Rightarrow$ i homo. equiv.

f homoto. equiv. $\Longleftrightarrow$ i homotop. equi $\Longleftrightarrow$ X is a deformation retract ∎

Attaching Spaces:

X_0, X_1, $A \subset X_1$ closed subspace, $f: A \longrightarrow X_0$ then

X_0 attached to X_1 along A via f $\leftarrow X_0 \sqcup_f X_1 = \dfrac{X_0 \sqcup X_1}{\{a \sim f(a) \mid a \in A\}}$

Mapping cylineder M_f for $f: X \to Y$ is the attached space of $X \times I$ to Y along $X \times \{1\}$ via f

$\underline{Q}$ If $f, g: A \longrightarrow X_0$ are homotopic, are $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$? No

$\underline{\text{Prop}}$: If (X_1, A) has HEP, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0.