Lecture 10

Wednesday, February 22, 2017 9:33 AM

Prop: Let X be path connected, locally path connected and semi locally simp. conn. top. space. Then for any subgroup $H < \pi_1(X, x_0)$, there exists a covering space $p: (\tilde{X}_H, \tilde{x}_0) \longmapsto (X, x_0)$ s.t. $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$.

Pf: **part I** Construct a simply connected covering space. (P.64)

Suppose $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a simply connected covering space.

- $\tilde{X}$ is simply connected $\Rightarrow$ for any $\tilde{x} \in \tilde{X}$ there is a unique homotopy class of path connecting $\tilde{x}_0$ to $\tilde{x}$.

$\Rightarrow$ any $\tilde{x} \in \tilde{X} \Longleftrightarrow$ A homotopy class of paths starting at $\tilde{x}_0$

$\underset{\text{HLP}}{\Longleftrightarrow}$ A homotopy class of paths starting at x_0.

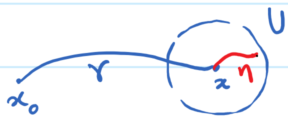
$$\tilde{X} = \{ [\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0 \} \qquad p: \tilde{X} \longrightarrow X, \quad [\gamma] \longrightarrow \gamma(1)$$

Basis for topology: $\mathcal{U} = \{ U \subset X \mid U \text{ is path connected, open}, \pi_1(U) \xrightarrow[\text{trivial}]{} \pi_1(X) \}$

- Ex: $\mathcal{U}$ is a basis of topology for X

For any $U \in \mathcal{U}$; and a path γ from x_0 to a pt in U

$$U_{[\gamma]} = \{ [\gamma \cdot \eta] \mid \eta \underset{\text{path}}{\subset} U \text{ s.t. } \eta(0) = \gamma(1) \}$$



- Ex: $\{U_{[\gamma]}\}$ form a basis for topology on $\tilde{X}$

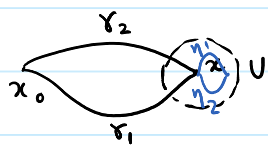
$p: \tilde{X} \longrightarrow X$ is a covering space. Let $x \in X$ and $U \in \mathcal{U}$ be an open nbd of x.

Then $p^{-1}(U) = \{ [\gamma] \mid \gamma \subset X \text{ is a path connecting } x_0 \text{ to a pt in } U \}$: is open

$\rightsquigarrow$ For each $[\gamma] \in p^{-1}(U)$, we have an open set $U_{[\gamma]} \subset p^{-1}(U)$.

$\Rightarrow$ p is conti.

$$p^{-1}(U) = \coprod U_{[\gamma]}$$ where $[\gamma]$ is a homotopy class of paths connecting x_0 to x.

① 

$\gamma_1 \not\simeq \gamma_2$, then if $[\gamma_1 \cdot \eta_1] = [\gamma_2 \cdot \eta_2]$ Then $[\gamma_1 \cdot \eta_1 \cdot \bar{\eta}_2 \cdot \bar{\gamma}_2] = 0$

$[\eta_1 \cdot \bar{\eta}_2] = 0 \rightsquigarrow [\gamma_1 \cdot \bar{\gamma}_2] = 0 \rightsquigarrow \gamma_1 \simeq \gamma_2$ ✗.

$\rightarrow U_{[\gamma_1]} \cap U_{[\gamma_2]} = \emptyset$

② $[\gamma'] \in p^{-1}(U)$, then take a path $\eta \subset U$ which connects $\gamma'(1)$ to x. Then $\gamma' \cdot \eta$ is a path connecting x_0 to x and $[\gamma'] \in U_{[\gamma' \cdot \eta]}$

• $p: U_{[\gamma]} \longrightarrow U$ is a homeo, because U path connected $\Rightarrow$ p surjective

• if $p([\gamma\cdot\eta_1]) = p([\gamma\cdot\eta_2]) \Rightarrow \eta_1(1) = \eta_2(1)$.

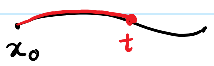
Since $\pi_1(U) \longrightarrow \pi_1(X)$ is trivial, $[\eta_1\bar{\eta}_2]$ is trivial.

$\Rightarrow \eta_1$ is homotopic to η_2 in $X \Rightarrow [\gamma\cdot\eta_1] = [\gamma\cdot\eta_2]$.

(Complete the details.)

* $\tilde{X}$ is simply connected.

① path connected : $[\gamma] \in \tilde{X}$



$$\gamma_t(s) = \begin{cases} \gamma(s) & 0 \leqslant s \leqslant t \\ \gamma(t) & 0 \leqslant t \leqslant 1 \end{cases}$$

$t \longmapsto [\gamma_t]$: path in $\tilde{X}$ connecting $[\gamma_0] = [x_0]$ to $[\gamma_1] = [\gamma]$.

$\rightsquigarrow$ path connected.

It's infact the lift of γ starting at $[x_0]$, because $p[\gamma_t] = \gamma(t)$.

$\rightsquigarrow$ For any $[\gamma] \in \pi_1(X, x_0)$ lift of γ starting at $[x_0]$ is a path connecting $[x_0]$ to $[\gamma]$. Thus if γ is null homotopic $\rightsquigarrow$ lift of γ is not a loop.

$\Rightarrow$ im(p_*) is the trivial subgroup of $\pi_1(X, x_0)$.

Part 2 H is not trivial. Let $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a simply connected covering space. H acts on $\tilde{X}$.

Def An action of a group G on a top. space Y is a homo

$$f: G \longrightarrow \text{homeo}(Y)$$

group of homeo of Y.

• $f(g): Y \longrightarrow Y$, usually instead of $f(g)(y)$ we write $g(y)$.

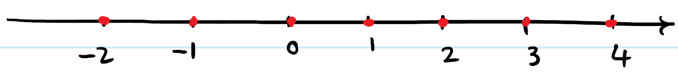
• For any $y \in Y$, $ey = y$, $(g_2 g_1)y = g_2(g_1(y))$

If G acts on Y, for any $y \in Y$ $G(y) = \{gy \mid g \in G\}$: orbit of y

$$Y/G = Y \big/ (y \sim g(y) \mid g \in G) \text{ : orbit space.}$$

This is an equivalence relation!

Ex $Y = \mathbb{R}$ $G = \mathbb{Z}$ $\mathbb{Z}$ acts on $\mathbb{R}$ by $x \xmapsto{n} x+n$



$G(x) = \{x, x\pm1, x\pm2, \dots\}$

$\mathbb{R}/\mathbb{Z}$: Circle

Ex $Y = \mathbb{R}\times\mathbb{R}$ $G = \mathbb{Z}\times\mathbb{Z}$ $(x,y) \xmapsto{(m,n)} (x+m, y+n) \Rightarrow Y/G$: Torus

An action of H on $\tilde{X}$: $[\gamma] \xrightarrow{[h]\in H} [h\cdot\gamma]$ (check that this is an action)

For $\tilde{x}=[\gamma]$ orbit of $\tilde{x} = \{ \overset{=[\gamma']}{[h\cdot\gamma]} \mid [h]\in H \}$

$= \{ [\gamma'] \mid \gamma(1)=\gamma'(1), [\gamma'\cdot\bar{\gamma}]\in H \}$.

$P:\tilde{X}\longrightarrow X$, $[\gamma]\longmapsto\gamma(1)$ induces a map $P_H:(\underset{\tilde{X}_H}{\underset{\parallel}{\tilde{X}/H}}, \tilde{x}_0) \longrightarrow (X,x_0)$ (*)

$\tilde{x}_0$: orbit of $[c_{x_0}]$: homotopy class of constant path at x_0

$(\tilde{X}_H,\tilde{x}_0)$ with the map P_H is the covering space.

• Take a nbd $x\in U\in\mathcal{U}$ of $x\in X$ as before. $P^{-1}(U)=\coprod_{\gamma(1)=x} U_{[\gamma]}$

If $P_H(U_{[\gamma]})\cap P_H(U_{[\gamma']})\neq\emptyset$, then $P_H(U_{[\gamma]})=P_H(U_{[\gamma']})$ as follows. Let

$[\gamma_1]\in U_{[\gamma]}$ and $[\gamma_2]\in U_{[\gamma']}$ s.t. $[\gamma_1]\sim[\gamma_2]$. Then $\gamma_1(1)=\gamma_2(1)$ and $[\gamma_1\bar{\gamma}_2]\in H$.

Thus for any $\eta\subset U$ s.t. $\eta(0)=\gamma_1(1)=\gamma_2(1)$ we have $[\gamma_1\cdot\eta]\sim[\gamma_2\cdot\eta]$.

$\Rightarrow P_H(U_{[\gamma_1]}=U_{[\gamma]}) = P_H(U_{[\gamma_2]}=U_{[\gamma']})$

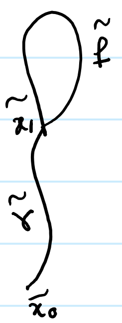
$\Rightarrow P_H:\tilde{X}_H\longrightarrow X$ covering map $\quad \overset{\text{quotient}}{q}:\tilde{X}\longrightarrow\tilde{X}_H$ covering map

• $P_{H*}(\pi_1(\tilde{X}_H,\tilde{x}_0))$: For any $\overset{\text{loop}}{\gamma}$ based at x_0 in X, lift of γ to $\tilde{X}_H$, starting

at $\tilde{x}_0$ is the image of the lifted path in $\tilde{X}$ under the quotient map q.

Lift of γ to $\tilde{X}$ is a path from $[c_{x_0}]$ to $[\gamma]$ and its image in $\tilde{X}_H$ is a loop iff

$[\gamma]\in H$. $\boxtimes$



Prop (a) $\overset{\text{path connected}}{P}:(\tilde{X},\tilde{x}_0)\longrightarrow(X,x_0)$ Then $\overset{H_0}{\overset{\parallel}{P_*(\pi_1(\tilde{X},\tilde{x}_0))}}$ and $\overset{H_1}{\overset{\parallel}{P_*(\pi_1(\tilde{X},\tilde{x}_1))}}$

$P:(\tilde{X},\tilde{x}_1)\longrightarrow(X,x_0)$ are conjugate subgroups of $\pi_1(X,x_0)$.

Pf : Let $\tilde{\gamma}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$. Then $P(\tilde{\gamma})$ is a loop based at x_0.

$\leadsto g=[P(\tilde{\gamma})]\in\pi_1(X,x_0) \leadsto g^{-1}H_0g=H_1$ because for any $[\tilde{f}]\in\pi_1(\tilde{X},\tilde{x}_1)$

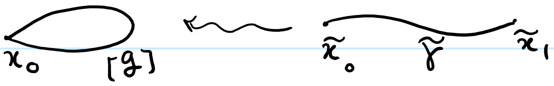
$[\tilde{\gamma}\tilde{f}\bar{\tilde{\gamma}}]\in\pi_1(\tilde{X},\tilde{x}_0)\Rightarrow gP_*[\tilde{f}]g^{-1}\in H_0 \leadsto gH_1g^{-1}\subset H_0$, similarly, $g^{-1}H_0g\subset H_1$

$\Rightarrow g^{-1} H_0 g = H_1$.

<u>Prop (b)</u> For $H_0 = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) < \pi_1(X, x_0)$ and $H_1 = g^{-1} H_0 g$ we have:

$H_1 = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$.

· $\tilde{x}_1 = \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the lift of a loop rep. g to $\tilde{X}$ starting at $\tilde{x}_0$.



Forgetting the base pt: for a path connected, locally path connected and semilocally sim.con space X, path connected covering space $\tilde{X}$

$\Updownarrow$

conjugacy classes of subgroups of $\pi_1(X)$.