Lecture 11

Sunday, February 26, 2017   8:12 PM

- <u>Part 1</u> : $H < \pi_1(X, x_0)$, Construct a covering space $P_H : (\tilde{X}_H, \tilde{x}_H) \to (X, x_0)$ such that $P_{H*}(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$

- <u>Part 2</u> Symmetries of a Covering Space (Deck trans. group)

<u>Recall</u> Define an action of H on $(\tilde{X}, \tilde{x}_0)$ where $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ universal cover
s.t. $p(\underset{H}{\underset{\cap}{h}} \cdot \tilde{x}) = p(\tilde{x})$ i.e. $\underset{H}{\underset{\cap}{h}} : p^{-1}(x) \to p^{-1}(x)$ for all $x \in X$.

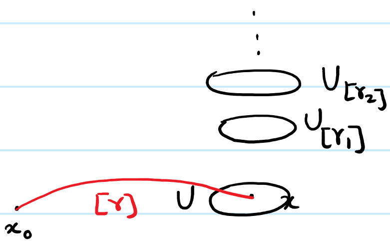
$H \ni h = [f]$, $\tilde{x} = [\gamma]$ $\Rightarrow$ $h \cdot \tilde{x} = [f \cdot \gamma]$

$p(\tilde{x}) = \gamma(1) = f \cdot \gamma(1) = p(h \cdot \tilde{x})$ ✓

<u>Equivalence relation</u> : $\tilde{x}_1 \sim \tilde{x}_2$ if for a $h \in H$ we have $h\tilde{x}_1 = \tilde{x}_2$.

Let $\tilde{X}_H = \tilde{X}/\sim$, $\tilde{x}_H = [\tilde{x}_0]$, P induces a map $P_H : (\tilde{X}_H, \tilde{x}_H) \to (X, x_0)$
$(P_H \circ q = P)$ quotient map

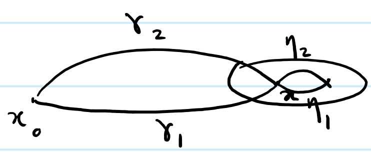
① P_H is a covering space map. For any $x \in X$, let U be a path connected open nbd s.t. $\pi_1(U) \to \pi_1(X)$ is trivial.



$$\Rightarrow p^{-1}(U) = \coprod_{\substack{\gamma(0) = x_0 \\ \gamma(1) = x}} U_{[\gamma]}$$

If for $[\gamma_1 \cdot \eta_1] \in U_{[\gamma_1]}$ and $[\gamma_2 \cdot \eta_2] \in U_{[\gamma_2]}$ we have $[\gamma_1 \cdot \eta_1] \sim [\gamma_2 \cdot \eta_2]$

$\Rightarrow [\gamma_1 \cdot \overbrace{\eta_1 \cdot \bar{\eta}_2}^{\text{trivial}} \cdot \bar{\gamma}_2] \in H \Rightarrow [\gamma_1 \cdot \bar{\gamma}_2] \in H \Rightarrow [\gamma_1] \sim [\gamma_2]$
and $[\gamma_1 \cdot \eta] \sim [\gamma_2 \cdot \eta]$



$\Rightarrow$ equivalence relation identifies $U_{[\gamma_1]}$ and $U_{[\gamma_2]}$ iff $[\gamma_1] \sim [\gamma_2] \Rightarrow P_H$ is covering map.

$\Rightarrow q : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}_H, \tilde{x}_H)$ is covering map.

② $P_{H*}(\pi_1(\tilde{X}_H, \tilde{x}_H))$: $P_H \, q = P$ ⇝ For a loop f based at x_0, lift of f to $\tilde{X}_H$ based at $\tilde{x}_H$ is $q\tilde{f} \Rightarrow q\tilde{f}(1) = \tilde{x}_0$ if $\tilde{f}(1) \sim \tilde{x}_0$ ⇝ $[f] \sim c_{x_0}$ ⇝ $[f] \in H$.

Def Action of a group G on a top space Y is called free if it has no fixed pt i.e.
$\forall y \in Y$ and $\forall g \in G$, $g \neq e$ $\Rightarrow$ $gy \neq y$.

- It's called a Covering space action, if for any $y \in Y$, there exists a nbd U of y such that for any $e \neq g \in G$, $gU \cap U = \emptyset$

- Covering space action $\Rightarrow$ free

Thm If action of G on a top space Y is a covering space action, then $q: Y \to Y/G$ is a Normal Covering Space.

Def A Covering space $p: \tilde{X} \to X$ is called normal if for any $x \in X$ and any $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$ there exists an isom of covering spaces $\tilde{X} \to \tilde{X}$ which takes $\tilde{x}_1$ to $\tilde{x}_2$.

- If $y_1 \sim y_2 \Rightarrow \exists\, g \in G$ s.t. $gy_1 = y_2$ $\rightsquigarrow$ isom. is homeo corresponding to g.

Def $p: \tilde{X} \to X$ covering space, group of covering space isom $\tilde{X} \to \tilde{X}$ is called deck trans. denoted $G(\tilde{X})$.

If $\tilde{X}$ normal $\Rightarrow$ $\tilde{X}/G(\tilde{X}) \approx X$ and $p: \tilde{X} \to \tilde{X}/G(\tilde{X})$

Suppose $\tilde{X}$ is path connected; uniq. lifting property implies that if such an f exists $\Rightarrow$ it's unique.

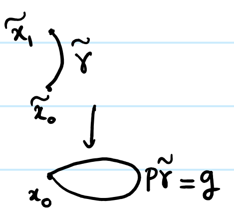
$$\begin{tikzcd}
(\tilde{X},\tilde{x}_1) \arrow[rr, "f"] \arrow[dr, "p"'] & & (\tilde{X},\tilde{x}_2) \arrow[dl, "p"] \\
& (X,x_0) &
\end{tikzcd}$$

$\Rightarrow$ Isom f is determined by where it sends one pt.

Cor If Y path connected, then $G(Y) \approx G$.

Thm Suppose X and $\tilde{X}$ are path connected and locally path connected. Then
$p: (\tilde{X},\tilde{x}_0) \to (X,x_0)$ normal $\iff$ $p_*(\pi_1(\tilde{X},\tilde{x}_0)) \triangleleft \pi_1(X,x_0)$ normal subgroup

Lem $p: (\tilde{X},\tilde{x}_0) \to (X,x_0)$ Then $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ and $p_*(\pi_1(\tilde{X},\tilde{x}_1))$ are conjugate.
$p: (\tilde{X},\tilde{x}_1) \to (X,x_0)$



$$[g]\, p_*(\pi_1(\tilde{X},\tilde{x}_1))\, [g]^{-1} = p_*(\pi_1(\tilde{X},\tilde{x}_0))$$

Pf $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ normal $\Leftarrow$ for any $\tilde{x}_1 \in p^{-1}(x_0)$ $p_*(\pi_1(\tilde{X},\tilde{x}_0)) = p_*(\pi_1(\tilde{X},\tilde{x}_1))$

$$\begin{tikzcd} (\tilde{X},\tilde{x}_0) \arrow[rr, shift left, "f"] \arrow[dr, "p"'] & & (\tilde{X},\tilde{x}_1) \arrow[ll, shift left, "g"] \arrow[dl, "p"] \\ & (X,x_0) & \end{tikzcd}$$

$\Leftrightarrow$ There exist a deck trans. which maps $\tilde{x}_0$ to $\tilde{x}_1$.

$\Leftrightarrow$ $\tilde{X}$ is normal.

Thm $G(\tilde{X}) \approx N(H)/H$ where $H = p_*(\pi_1(\tilde{X},\tilde{x}_0))$ and $N(H)$ is normalizer of H in $\pi_1(X,x_0)$.

Pf Construct homo $\varphi: N(H) \longrightarrow G(\tilde{X})$

$[\gamma] \xrightarrow[\text{surjective}]{} \tau$ $\leftarrow$ isom which maps $\tilde{x}_0$ to $\tilde{x}_1 = \tilde{\gamma}(1)$.

homo : $\varphi([\gamma\gamma']) = \tilde{x}_0 \longmapsto \widetilde{\gamma\gamma'}(1) = \tau\tilde{\gamma}'(1) = \tau\tau'(\tilde{x}_0) \Rightarrow \varphi([\gamma\gamma']) = \tau\tau' = \varphi([\gamma])\varphi([\gamma'])$

$[\gamma] \in Ker(\varphi)$ if γ lifts to a loop in $\tilde{X}$ $\Leftrightarrow$ $[\gamma] \in H$ $\Rightarrow$ $Ker(\varphi) = H$

Cor If $\tilde{X}$ is normal then $G(\tilde{X}) \approx \pi_1(X,x_0)/H$

Cor If $\tilde{X}$ is the universal cover of X, $G(\tilde{X}) \approx \pi_1(X,x_0)$.

Ex $p: \mathbb{R} \to S^1$ $\quad G(\mathbb{R}) \approx \mathbb{Z}$ $\quad$ | $\quad \mathbb{R}\times\mathbb{R} \longmapsto T^2$

$x \xmapsto{m\in\mathbb{Z}} x+m$ $\quad$ | $\quad G(\mathbb{R}\times\mathbb{R}) \approx \mathbb{Z}\times\mathbb{Z}$ $\quad (x,y) \xmapsto{(m,n)} (x+m, y+n)$

Cor If Y path connected, locally path connected, $G \approx \pi_1(Y/G)/p_*(\pi_1(Y))$.

$(q: Y \to Y/G)$