Lecture 12

Tuesday, February 28, 2017 8:45 PM

Simplicial homology:

* Fundamental group is not good for detecting high-dimensional space
    e.g. $\pi_1(S^n) = 0 \quad n \geq 2$
    - π_1 of any CW Complex X is determined by X^2

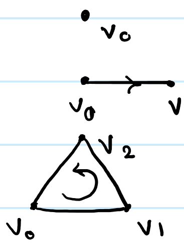
  $\Rightarrow$ Higher homotopy group $\leftarrow$ not easy to Calculate
* Homology is an easier invariant to calculate

Def An <u>n-simplex</u> is the Convex hull of $n+1$ affinely independent pts $v_0, \dots, v_n \in \mathbb{R}^N$ i.e. vectors $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ are linearly indep.

0-Simplex

1-Simplex

2-Simplex



denoted by $[v_0, v_1, \dots, v_n]$

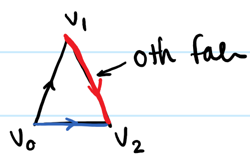
Standard n-simplex $\Delta^n = [\underset{v_0}{(1,0,\dots,0)}, \underset{v_1}{(0,1,0\dots,0)}, \dots, \underset{v_n}{(0,0,\dots,1)}]$ (in $\mathbb{R}^{n+1}$)

$$= \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} t_i = 1,\ t_i \geq 0 \text{ for all } i\}$$

Canonical linear homeo: $\Delta^n \longmapsto [v_0, \dots, v_n]$

$$(t_0, t_1, \dots, t_n) \longmapsto \sum_{i=1}^{n} t_i v_i$$

Def $[v_0, \dots, v_n] \longrightarrow$ ith face $[v_0, v_1, \dots, \hat{v}_i, \dots, v_n] = [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$



Def A <u>Δ-Complex</u> str on a top. space X is a collection of maps $\sigma_\alpha : \Delta^n \longrightarrow X$ for each n s.t.

(1) $\sigma_\alpha|_{\mathring{\Delta}^n}$ is inj and each pt of X is in the image of exactly one $\sigma_\alpha|_{\mathring{\Delta}^n}$
($\mathring{\Delta}^n$ = interior of Δ^n)

(2) For each i,

$$\begin{array}{ccc} [v_0, \dots, \hat{v}_i, \dots, v_n] & \xleftarrow[\text{iden.}]{\text{Canonical}} & \Delta^{n-1} \\ \downarrow \sigma_\alpha & \swarrow \sigma_\beta & \\ X & & \end{array}$$

σ_β ← one of the maps in the Δ-complex str.

(3) $A \subset X$ open $\iff$ $\sigma_\alpha^{-1}(A)$ open in Δ^n for each σ_α.

In other word, X is Construct as the quotient space

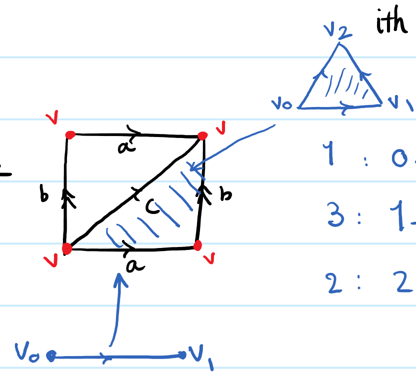
$$X = \frac{\coprod_n \coprod_\alpha \Delta^n_\alpha}{d_i \Delta^n_\alpha \sim \Delta^{n-1}_\beta}$$

Δ^n_α ← take one simplex Δ^n_α for $\sigma_\alpha : \Delta^n \to X$

$d_i \Delta^n_\alpha$: ith face of Δ^n_α

Δ^{n-1}_β ← $\sigma_\beta : \Delta^{n-1} \to X$ is the $(n-1)$-simplex Corresponding to $\sigma_\alpha \big|_{\text{ith face of } \Delta^n}$

EX



1 : 0-Simplex

3 : 1-Simplex

2 : 2-simplex

X : Δ-Complex $\Rightarrow$ $S_n(X)$: the set of n-simplicies $\{\sigma_\alpha : \Delta^n \to X\}_\alpha$

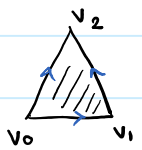
$\Delta_n(X)$: free abelian group gen. by the elements of $S_n(X)$

elements of the form $\sum_{\sigma_\alpha \in S_n(X)} a_\alpha \sigma_\alpha$ $\quad a_\alpha \in \mathbb{Z}$ and nonzero for finitely many α's

Boundry homo : $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$

$$\partial_n \left(\sum a_\alpha \sigma_\alpha \right) = \sum a_\alpha \partial_n \sigma_\alpha$$

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha \big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$



$n=2$ $\quad \partial(\sigma) = \sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]} + \sigma|_{[v_0, v_1]}$

Lem $\quad \partial_{n-1} \circ \partial_n = 0$

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$$\partial_{n-1}\partial_n(\sigma) = \sum_i (-1)^i \left(\sum_{j<i} (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]} + \sum_{j>i} (-1)^{j-1} \sigma|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]} \right)$$

$$= \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]} + \sum_{j>i} (-1)^{i+j-1} \sigma|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]} = 0$$

$\Rightarrow$ For any n $\quad \partial_n \circ \partial_{n+1} = 0 \Rightarrow \operatorname{Ker}(\partial_n) \supset \operatorname{Im}(\sigma_{n+1})$

Def . Elements of $\ker \partial_n$ are called n-cycles : $Z_n^\Delta = \ker(\partial_n)$

. Elements of $\operatorname{Im}(\partial_{n+1})$ are called n-boundaries : $B_n^\Delta = \operatorname{Im}(\partial_{n+1})$

n-Simplicial homology group of X : $H_n^\Delta(X) = Z_n^\Delta / B_n^\Delta$

Sequence $\rightsquigarrow$

$$\Delta_n(X) \boxed{\xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}}} \Delta_{n-2}(X) \longrightarrow \cdots \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0$$

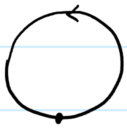
$$H_{n-1}^\Delta(X) = \ker \partial_{n-1} / \operatorname{Im} \partial_n$$

Ex 1 $X = \{pt\}$ $\Delta_i(X) = 0 \quad i > 0$

$\Delta_0(X) = \mathbb{Z}$

$\rightsquigarrow \; 0 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0 \rightsquigarrow H_i^\Delta(X) = \begin{cases} 0 & i \geq 1 \\ \mathbb{Z} & i = 0 \end{cases}$

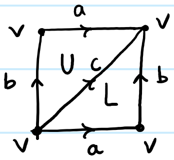
Ex 2 $X = S^1$



$\Rightarrow \partial e = v - v = 0 \Rightarrow \partial_1 = 0$

$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_1 = 0} \mathbb{Z} \xrightarrow{\partial_0 = 0} 0 \Rightarrow H_i^\Delta(S^1) = \begin{cases} \mathbb{Z} & i = 0,1 \\ 0 & i \geq 2 \end{cases}$

Ex 3 $X = T$



$\Rightarrow \partial a = 0, \; \partial b = 0, \; \partial c = 0 \Rightarrow \partial_1 = 0$

$\partial U = \boxed{a + b - c}$

$\partial L = a + b - c$

$\Rightarrow \partial(\underset{\sim}{U - L}) = 0$

generates the ker of ∂_2

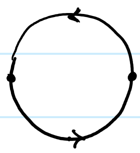
$$0 \longrightarrow \underset{U}{\mathbb{Z}} \oplus \underset{L}{\mathbb{Z}} \xrightarrow{\partial_2} \overset{a}{\mathbb{Z}} \oplus \overset{b}{\mathbb{Z}} \oplus \overset{c}{\mathbb{Z}} \xrightarrow{\partial_1 = 0} \overset{v}{\mathbb{Z}} \xrightarrow{\partial_0} 0$$

$a = a + b - c - b + c$

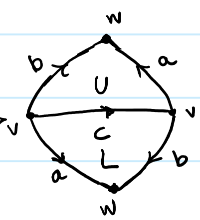
$b, \; c, \; a + b - c$

$\Rightarrow H_i^\Delta(T) = \begin{cases} 0 & i > 2 \\ \mathbb{Z} & i = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 0 \end{cases}$

EX $X = \mathbb{R}P^2$



$\rightsquigarrow$



$\Leftarrow$ Δ-Complex str.

$\partial_1 a = w - v \quad \partial_1 b = w - v \quad \partial_1 c = 0 \rightsquigarrow \ker \partial_1 =$ gen by $\{b - a, c\}$ = gen by $\{c + b - a, c\}$

$\operatorname{Im} \partial_1 =$ gen. by $\{w - v\} \rightsquigarrow H_0^\Delta(\mathbb{R}P^2) = \mathbb{Z}$

$\partial_2 U = c + a - b = c - (b - a)$

$\partial_2 L = c + b - a$

linearly indep $\rightsquigarrow \ker \partial_2 = 0 \rightsquigarrow H_2^\Delta(\mathbb{R}P^2) = 0$

$\rightsquigarrow \operatorname{Im} \partial_2 :$ gen by $\{c + b - a, \; c + a - b\}$ = gen by $\{c + b - a, 2c\}$

$\Rightarrow H_1^\Delta(\mathbb{R}P^2) = \mathbb{Z}_2$

$$0 \longrightarrow \overset{U}{\mathbb{Z}} \oplus \overset{U+L}{\mathbb{Z}} \xrightarrow{\partial_2} \overset{c}{\mathbb{Z}} \oplus \overset{c+b-a}{\mathbb{Z}} \oplus \overset{a}{\mathbb{Z}} \xrightarrow{\partial_1} \overset{v}{\mathbb{Z}} \oplus \overset{w-v}{\mathbb{Z}} \xrightarrow{\partial_0} 0$$