Lecture 13

Sunday, March 5, 2017 9:55 PM

Singular homology

Def A chain complex $(C_*, \partial_*) = \{(C_n, \partial_n)\}$ is a sequence

$$\longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where each C_i is an abelian group and each ∂_i is a homo such that

$$\partial_n \partial_{n+1} = 0 \quad \text{for any } n.$$

$\Rightarrow$ Define $H_n = \dfrac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$: n-th homology group, elements of C_n : n-chain

Simplicial homology : Δ-complex $X \Rightarrow$ chain complex $(C_n^{\Delta}(X), \partial_n)$ and defined $H_n^{\Delta}(X)$ to be the n-th homology group of the chain complex.

Singular homology :

X : top space

Def A singular n-simplex in X is a map $\sigma : \Delta^n \longrightarrow X$.

Def $S_n(X)$: set of singular n-simplicies in X

$C_n(X)$: free abelian group gen. by the elements of $S_n(X)$.

$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma\big|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$$

The same proof as before implies that $\partial_n \partial_{n+1} = 0$

$\Rightarrow$ chain complex: $\longrightarrow \boxed{C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)} \longrightarrow \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0 = 0} 0$

$H_n(X) = \operatorname{Ker}(\partial_n) / \operatorname{Im}(\partial_{n+1})$

Ex $X = \{pt\}$ $\quad C_n(X) \approx \mathbb{Z}$ gen. by constant map $\sigma_n : \Delta^n \longrightarrow \{pt\}$

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \, \sigma_n\big|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]} = \sum_{i=0}^{n} (-1)^i \, \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \Rightarrow \partial_n = 0 \\ \sigma_{n-1} & n \text{ even} \Rightarrow \partial_n : \text{isom} \end{cases}$$

$\Rightarrow \quad \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \cdots \boxed{\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0}$ $\quad H_n(X) = 0$ for $n \geq 1$.

$H_0(X) \approx \mathbb{Z}$

Compare Simplicial and Singular homology

① Simplicial homology is more rigid comparing to Singular homology.

Q Does $H_n^{\Delta}(X)$ indep. of the Δ-Complex str ?

Singular homology is defined for any top. space and from the def you can see that it's inv. under homeo.

② $C_n(X)$ is very large, usually uncountably many elements

- From the def is not obvious if for a Δ-Complex with no n-simplex for $n \geq N$
  $H_n(X) = 0$ for $n \geq N$ while $H_n^{\Delta}(X) = 0$ for $n \geq N$.

③ Singular homology seems more general than simplicial homology, but

$$X \rightsquigarrow \underbrace{S(X)}_{\text{top space with a } \Delta\text{-Complex str.}} \quad \text{s.t. } H_n(X) \approx H_n^{\Delta}(S(X)).$$

Prop If $X = \bigcup_\alpha X_\alpha$ where each X_α is a path connected component of X, then $H_n(X) \approx \bigoplus_n H_n(X_\alpha)$.

pf

$\sigma: \Delta^n \to X \rightsquigarrow \mathrm{im}(\sigma) \subset X_\alpha$ for some $\alpha \Rightarrow \sigma \in C_n(X_\alpha) \Rightarrow C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$

If $\sigma \in C_n(X_\alpha) \rightsquigarrow \partial_n \sigma \in C_{n-1}(X_\alpha) \Rightarrow \partial_n$ preserves decom.

$\Rightarrow H_n(X) \approx \bigoplus_n H_n(X_\alpha)$.

Prop If X is path-connected, $H_0(X) \approx \mathbb{Z}$.

pf $\cdots C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0 = 0} 0 \rightsquigarrow H_0(X) = C_0(X) / \mathrm{Im}\,\partial_1$

Basis idea

0-simplex $\longleftrightarrow$ pt $\in X$,

σ_0, σ_0' ($\sigma_0 \wr p$, $\sigma_0' \wr p'$) $\Rightarrow \gamma: I \xrightarrow{\text{"}\Delta^1\text{"}} X$, $\gamma(0) = p$, $\gamma(1) = p'$

$\Rightarrow \partial\gamma = \underbrace{\sigma_0' - \sigma_0}_{\in \mathrm{Im}\,\partial_1}$

$\Rightarrow \sigma_0 \sim \sigma_0'$

$\varepsilon: C_0(X) \longrightarrow \mathbb{Z}$

$\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$

- ε is surjective $\Rightarrow \varepsilon(n\sigma) = n$
- $\mathrm{Ker}(\varepsilon) = \mathrm{Im}\,\partial_1$.

  $\mathrm{Im}(\partial_1) \subset \mathrm{Ker}(\varepsilon)$ because $\partial_1(\sigma) = \sigma|_{[v_1]} - \sigma|_{[v_0]} \Rightarrow \varepsilon(\partial_1 \sigma_1) = 0$

  $\varepsilon(\sum n_i \sigma_i) = 0 \Rightarrow$ For any x_i consider a path τ_i connecting x_0 to x_i.

  $\mathrm{Im}(\sigma_i) = x_i \quad \Rightarrow \partial(\sum_i n_i \tau_i) = \sum_i n_i \sigma_i - (\sum_i n_i)\sigma_0 = \sum_i n_i \sigma_i$

Cor $X = \bigcup_\alpha X_\alpha$ ← path connected comp. $\Rightarrow H_0(X) \approx \bigoplus_\alpha \mathbb{Z}$

Def Reduced homology groups of X; denoted by $\tilde{H}_n(X)$:
(X is path connected)

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

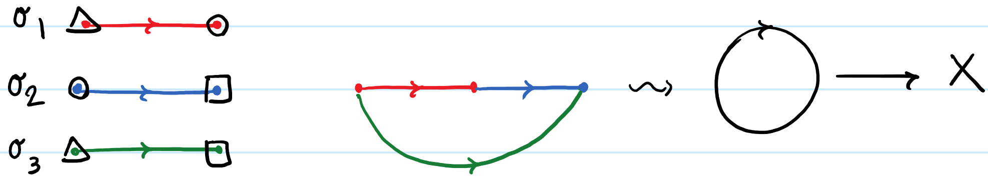
$$\tilde{H}_n(X) \approx \begin{cases} H_n(X) & n \geq 1 \\ 0 & n = 0 \end{cases}$$

Geometric picture

Recall: Elements of $Z_n(X) = \mathrm{Ker}(\partial_n)$ are called n-cycles & elements of $B_n(X) = \mathrm{Im}(\partial_{n+1})$ are called n-boundries

For $n=1$, Consider $\sum_i n_i \sigma_i \in Z_n(X) \longleftrightarrow \coprod_\alpha S^1_\alpha \longrightarrow X$

- For example, $\sigma_1 + \sigma_2 - \sigma_3 \in Z_n(X) \rightsquigarrow \sigma_1|_{[v_1]} - \sigma_1|_{[v_0]} + \sigma_2|_{[v_1]} - \sigma_2|_{[v_0]} - (\sigma_3|_{[v_1]} - \sigma_3|_{[v_0]}) = 0$



Similarly, if $\underbrace{\sum_i n_i \sigma_i}_{\partial_2 \xi} \in B_n(X) \Rightarrow$ Corresponding to ξ we get a map from a disjoint union of oriented surfaces to X such that the restriction of the map to the boundry gives the oriented loops associated with $\sum_i n_i \sigma_i$.