Lecture 14

Tuesday, March 7, 2017 9:12 PM

Def C_* and D_* are chain complexes. A chain map $\varphi : C_* \longrightarrow D_*$ is a sequence of homo $\varphi_n : C_n \longrightarrow D_n$ s.t. $\varphi_{n-1}\partial_n = \partial_n \varphi_n$ i.e. diagram

$$\begin{array}{ccccccccccc}
\longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & C_{n-2} & \xrightarrow{\partial_{n-2}} & \cdots & \xrightarrow{\partial_1} & C_0 & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_{n-2}} & & & & \downarrow{\scriptstyle \varphi_0} & \\
\longrightarrow & D_n & \xrightarrow{\partial_n} & D_{n-1} & \xrightarrow{\partial_{n-1}} & D_{n-2} & \xrightarrow{\partial_{n-2}} & \cdots & \xrightarrow{\partial} & D_0 & \longrightarrow 0
\end{array}$$

commutes.

Lem A chain map $\varphi : C_* \longrightarrow D_*$ induces a homo $\varphi_* : H_n(C_*) \longrightarrow H_n(D_*)$ for any n.

$$H_n(C_*) = \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}}$$

Pf

If $\alpha \in C_n$, $\partial_n \alpha = 0 \Rightarrow \varphi_{n-1}\partial_n \alpha = 0 \rightsquigarrow \partial_n \varphi_n \alpha = 0 \rightsquigarrow \varphi_n \alpha$ is an n-cycles.

$\alpha = \partial_{n+1}\beta \Rightarrow \varphi_n \alpha = \varphi_n(\partial_{n+1}\beta) = \partial_n \varphi_{n+1}\beta \Rightarrow \varphi_n \alpha$ is boundry.

$\Rightarrow \varphi_*([\alpha]) = [\varphi(\alpha)]$ is well defined.

Properties
- $C_* \xrightarrow{\varphi} D_* \xrightarrow{\psi} E_*$ then $(\psi\varphi)_* = \psi_* \varphi_*$
- $C_* \xrightarrow{\mathbb{1}} C_*$ then $\mathbb{1}_* = \mathbb{1}$.

Let X and Y be top. spaces and $f : X \longrightarrow Y$ be a conti. map. Then f induces homos $f_\# : C_n(X) \longrightarrow C_n(Y)$ by composition i.e.

$$\sigma : \Delta^n \longrightarrow X \quad \Longrightarrow \quad f_\#(\sigma) = f\sigma : \Delta^n \longrightarrow Y$$

Thus: $f_\#\big(\underbrace{\sum_i n_i \sigma_i}_{\in\, C_n(X)}\big) = \sum_i n_i f_\#(\sigma_i)$

Lem $f_\#$ is a chain map.

$$\sigma : \Delta^n \longrightarrow X \rightsquigarrow f_\# \partial\sigma = f_\#\Big(\sum_i (-1)^i \sigma\big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}\Big) = \sum_i (-1)^i f\sigma\big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$$= \partial f_\#(\sigma)$$

Cor f induces a homo $f_* : H_n(X) \longrightarrow H_n(Y)$ for any n.

Thm If $f, g : X \to Y$ are homotopic, then $f_* = g_*$.

Def Suppose $\varphi, \psi : C_* \to D_*$ be chain maps. Then φ and ψ are called chain homotopic if there exists a seq. of homos. $P_n : C_n \to D_{n+1}$ such that $\psi_n - \varphi_n = \partial_{n+1} P_n + P_{n-1} \partial_n$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\psi_{n+1}-\varphi_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\psi_n-\varphi_n"] \arrow[dl, "P_n"'] & C_{n-1} \arrow[r] \arrow[d, "\psi_{n-1}-\varphi_{n-1}"] \arrow[dl, "P_{n-1}"'] & \cdots \\
\arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"'] & D_n \arrow[r, "\partial_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Lem If $\varphi, \psi : C_* \to D_*$ are chain homotopic, then $\varphi_* = \psi_* : H_n(C) \to H_n(D)$.

Pf $\alpha \in C_n$ s.t. $\partial_n \alpha = 0 \Rightarrow \psi_n(\alpha) - \varphi_n(\alpha) = \partial_{n+1} P_n(\alpha) \in \mathrm{Im}(\partial_{n+1})$

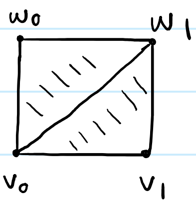
$\Rightarrow [\psi_n(\alpha)] = [\varphi_n(\alpha)] \Rightarrow \varphi_* = \psi_*$.

Lem If $f, g : X \to Y$ are homotopic, then $f_\#$ and $g_\#$ are chain homotopic.

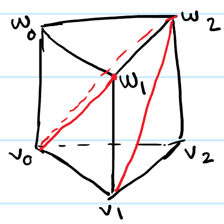
pf Let $F : X \times I \to Y$ be the homotopy from f to g. Define a chain homotopy $P_n : C_n(X) \to C_{n+1}(Y)$ as follows.

Let $\sigma : \Delta^n \to X$ be a singular n-simplex. Set

$\bar{\sigma} = F \circ (\sigma \times \mathbb{1}) : \Delta^n \times I \to X \times I \xrightarrow{F} Y$



$[v_0, v_1, w_1] \cup [v_0, w_0, w_1]$



$[v_0, v_1, v_2, w_2]$
$\cup$
$[v_0, v_1, w_1, w_2]$
$\cup$
$[v_0, w_0, w_1, w_2]$

$$P_n(\sigma) = \sum_i (-1)^i \bar{\sigma}\big|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$$

$$\partial P_n(\sigma) = \sum_i (-1)^i \partial \bar{\sigma}\big|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]} = \sum_i (-1)^i \sum_{j=0}^{i-1} (-1)^j \bar{\sigma}\big|_{[v_0, \ldots \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n]}$$

$$+ \underbrace{\sum_i \bar{\sigma}\big|_{[v_0, \ldots, v_{i-1}, w_i, \ldots, w_n]} - \sum_i \bar{\sigma}\big|_{[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]}}_{\bar{\sigma}|_{[w_0, \ldots, w_n]} - \bar{\sigma}|_{[v_0, \ldots, v_n]}} + \sum_i (-1)^i \sum_{j=i+1}^{n} (-1)^{j+1} \bar{\sigma}\big|_{[v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n]}$$

$$= \underbrace{\sum_{j<i} (-1)^{i+j} \bar{\sigma}\Big|_{[v_0,\dots,\hat{v}_j,\dots v_i, w_i,\dots,w_n]} - \sum_{j>i} (-1)^{i+j} \bar{\sigma}\Big|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}}_{\overset{\shortparallel}{-P_{n-1}(\partial\sigma)}} + \underbrace{\overset{F_0(\sigma\times\mathbb{1})}{\overset{\shortparallel}{\bar{\sigma}}}\Big|_{[w_0,\dots,w_n]}}_{g\circ\sigma = g_\#(\sigma)} - \underbrace{\bar{\sigma}\Big|_{[v_0,\dots,v_n]}}_{\overset{\shortparallel}{f_\#\sigma}}$$

$$\partial\sigma = \sum_i (-1)^i \sigma\Big|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

$$\bar{\sigma}_i = F_0(\sigma\Big|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} \times \mathbb{1}) = F_0(\sigma\times\mathbb{1})\Big|_{[v_0,\dots,\hat{v}_i,\dots,v_n]\times I}$$

n-simplicies here are $j<i$ $[v_0,\dots,v_j,w_j,\dots,\hat{w}_i,\dots,w_n]$

$j>i$ $[v_0,\dots,\hat{v}_i,\dots,v_j,w_j,\dots,w_n]$

$$\Rightarrow P_{n-1}(\partial\sigma) = \sum_{j<i} (-1)^{i+j} \bar{\sigma}\Big|_{[v_0,\dots,v_j,w_j,\dots,\hat{w}_i,\dots,w_n]} + \sum_{j>i} (-1)^{i+j-1} \bar{\sigma}\Big|_{[v_0,\dots,\hat{v}_i,\dots,v_j,w_j,\dots,w_n]}$$

$\Rightarrow \partial P_n + P_{n-1}\partial = g_\# - f_\# \Rightarrow g_\#$ and $f_\#$ are chain homotopic. ▨

<u>Cor</u> If $f: X \to Y$ is a homotopy equivalence, then $f_*: H_n(X) \to H_n(Y)$ is isom.

<u>pf</u> $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ $\quad fg \simeq \mathbb{1} \Rightarrow f_* g_* = \mathbb{1} \Rightarrow f_*$ is isom.

$gf \simeq \mathbb{1} \Rightarrow g_* f_* = \mathbb{1}$

<u>Cor</u> If X is <u>contractible</u>, $X \simeq \{pt\} \Rightarrow H_n(X) = \begin{cases} 0 & n \geq 1 \\ \mathbb{Z} & n = 0 \end{cases}$

<u>Exact sequences and relative homology</u>

- $A \underset{\text{subspace}}{\subset} X \Rightarrow C_n(A) \subseteq C_n(X) \rightsquigarrow C_n(X,A) = C_n(X)/C_n(A)$

- ∂_n induces a homo $\partial_n: C_n(X,A) \longrightarrow C_n(X,A)$

  $\partial_n[\alpha] = [\partial_n \alpha]$, $\alpha - \beta \in C_n(A) \Rightarrow \partial_n(\alpha-\beta) \in C_{n-1}(A)$

  $\Rightarrow [\partial_n\alpha] = [\partial_n\beta]$

- $\partial_n \circ \partial_{n+1} = 0$ because $\partial_n \circ \partial_{n+1}[\alpha] = \partial_n \circ [\partial_{n+1}\alpha] = [\partial_n\partial_{n+1}\alpha] = 0$

$\Rightarrow$ Chain Complex: $\longrightarrow C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \dots$

$\Rightarrow$ Homology groups of this chain Complex are called <u>relative homology groups</u>. $H_n(X,A)$.

<u>Def</u> • A <u>relative n-cycle</u>, n-chain $\alpha \in C_n(X)$ s.t. $\partial_n(\alpha) \in C_{n-1}(A)$.

- A <u>relative n-boundry</u> is a relative n-cycle $\alpha \in C_n(X,A)$ such that $\alpha = \partial\beta + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$

  $([\alpha] = \partial_{n+1}[\beta] \Rightarrow [\alpha] = [\partial_{n+1}\beta] \Rightarrow \alpha - \partial\beta \in C_n(A))$

Plan Relate the homology groups of A , X and relative homology groups (X, A).

Def An exact sequence is a sequ. of homo

$$\cdots \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \cdots$$

such that $\ker(\alpha_n) = \operatorname{Im}(\alpha_{n+1})$ for all n.

Short exact sequence : $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$

$\Rightarrow$ α : inj. β: surjective and $\ker(\beta) = \operatorname{Im}\alpha$

$\Rightarrow$ $C \approx B/\operatorname{Im}\alpha \approx B/A$

Ex $0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \to 0$