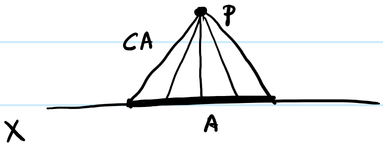
Lecture 16

Monday, March 27, 2017 9:12 AM

Goal
- Excision thm
- Corollaries
- Equivalence of singular and Simplicial homology

Excision thm If for subspaces $Z \subset A \subset X$ we have $\bar{Z} \subset \operatorname{int}(A)$, then $\iota_* : H_n(X-Z, A-Z) \longrightarrow H_n(X, A)$ induced by the inclusion map ι is an isomorphism.

Cor 1 $H_n(X, A) \approx \tilde{H}_n(\underbrace{X \cup CA}_{\substack{\Downarrow \\ \text{mapping Cone of } i : A \hookrightarrow X}})$ where $X \cup CA = \dfrac{X \amalg A \times I}{\big((a,0) \sim (a',0),\ (a,1) \sim \iota(a) \mid a, a' \in A\big)}$



Pf $CA \subset X \cup CA$

$$\cdots \longrightarrow \cancelto{0}{\tilde{H}_n(CA)} \longrightarrow \tilde{H}_n(X \cup CA) \longrightarrow H_n(X \cup CA, CA) \longrightarrow \cancelto{0}{\tilde{H}_{n-1}(CA)} \longrightarrow \cdots$$

$\Rightarrow \tilde{H}_n(X \cup CA) \approx H_n(X \cup CA, CA)$

$\{P\} \subset CA \subset X \cup CA \Rightarrow H_n(X \cup CA, CA) \approx H_n(X \cup CA - \{P\}, CA - \{P\})$

$\approx H_n(X, A)$

Recall from chapter 0: $CA \subset X \cup CA$ is a Contractible subspace

$\Rightarrow$ if CA has HEP, then $X \cup CA \simeq X/A \Rightarrow \tilde{H}_n(X \cup CA) \simeq \tilde{H}_n(X/A)$

$\Updownarrow$

(it has an appropriate mapping cylinder nbd)

Def (X, A) is called good, if A is closed and a deformation retraction of a nbd $A \subset U$ in X. (For example, X is a CW complex and $A \subset X$ is a subcomplex)

In particular, (X, A) good $\Rightarrow$ CA has HEP $\Rightarrow H_n(X, A) \approx \tilde{H}_n(X/A)$

In fact : $H_n(X, A) \xrightarrow{q_*} H_n(X/A, A/A) \approx \tilde{H}_n(X/A)$ is an isom. $\rightsquigarrow$

$$\begin{tikzcd}
H_n(X \cup CA, CA) \arrow[rr, leftrightarrow, "i_*", "\approx"'] \arrow[dr, "q_*", "\approx"'] & \arrow[d, phantom, "\circlearrowleft"] & H_n(X, A) \arrow[dl, "q_*"'] \\
& H_n(X/A, A/A) &
\end{tikzcd}$$

$\Rightarrow$ Exact Seq: $\cdots \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X/A) \longrightarrow \tilde{H}_{n-1}(A) \longrightarrow \cdots$

Ex Let $X = D^n$, $A = S^{n-1}$

$$\begin{tikzcd}
{} \arrow[r] & \tilde{H}_k(S^{n-1}) \arrow[r] & \cancel{\tilde{H}_k(D^n)} \arrow[r] \arrow[d] & \tilde{H}_k(\underbrace{D^n/S^{n-1}}_{\substack{\cong \\ S^n}}) \arrow[r] & \tilde{H}_{k-1}(S^{n-1}) \arrow[r] & \cancel{\tilde{H}_{k-1}(D^n)} \arrow[r] \arrow[d] & \cdots \\
 & & 0 & & & 0 &
\end{tikzcd}$$

$$\Rightarrow \tilde{H}_k(S^n) \approx \tilde{H}_{k-1}(S^{n-1}) \approx \tilde{H}_{k-2}(S^{n-2}) \approx \cdots \approx \tilde{H}_0(S^{n-k}) = \begin{cases} 0 & n>k \\ \mathbb{Z} & n=k \end{cases} \quad (n \geqslant k)$$

For $k>n$ $\rightsquigarrow$ $\tilde{H}_k(S^n) \approx \tilde{H}_{k-n}(S^0) = 0$

$$\Rightarrow \tilde{H}_k(S^n) \approx \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k=n \end{cases}$$

Cor 2 (Wedge sum) Suppose (X_α, x_α) is a based top. space s.t. $(X_\alpha, \{x_\alpha\})$ is good.

$$\Rightarrow \quad \tilde{H}_n(\vee_\alpha X_\alpha) \approx \oplus_\alpha \tilde{H}_n(X_\alpha)$$

Pf $X = \coprod_\alpha X_\alpha$ $\quad A = \coprod_\alpha \{x_\alpha\} \Rightarrow X/A \simeq \vee_\alpha X_\alpha$

$$\Rightarrow \begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_n(A) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \tilde{H}_n(X) \arrow[r, "\approx"] \arrow[d, phantom, "\shortparallel"] & \tilde{H}_n(\vee_\alpha X_\alpha) \arrow[r] & \tilde{H}_{n-1}(A) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \cdots \\
 & 0 & \oplus_\alpha \tilde{H}_n(X_\alpha) & & 0 &
\end{tikzcd} \quad (n>1)$$

Ex $\tilde{H}_1(\vee_n S^1) \approx \oplus_n \mathbb{Z}$

Def Let X be a top. space s.t. each pt $x \in X$ is closed. Then $H_n(X, X-\{x\})$ are called local homology groups of X at x.

(Because. let $\{x\} \in U$ be an open set, then set $Z = X-U$ $\Rightarrow$

$H_n(U, U-\{x\}) \approx H_n(X, X-\{x\})$ $\leftsquigarrow$ depends on top around x)

Cor3 $\mathbb{R}^n$ is homeo to $\mathbb{R}^m$ iff $m=n$.

Compute $H_k(\mathbb{R}^n, \mathbb{R}^n - \{0\})$

$$\begin{tikzcd}
\cdots \arrow[r] & \cancel{\tilde{H}_k(\mathbb{R}^n)} \arrow[r] & H_k(\mathbb{R}^n, \mathbb{R}^n-\{0\}) \arrow[r] & \tilde{H}_{k-1}(\mathbb{R}^n-\{0\}) \arrow[r] & \cancel{\tilde{H}_{k-1}(\mathbb{R}^n)} \arrow[r] \arrow[d] & \cdots \\
 & 0 & & & 0 &
\end{tikzcd}$$

$$\Rightarrow H_k(\mathbb{R}^n, \mathbb{R}^n-\{0\}) \approx \tilde{H}_{k-1}(\mathbb{R}^n-\{0\}) \approx \tilde{H}_{k-1}(S^{n-1}) \approx \begin{cases} \mathbb{Z} & k=n \\ 0 & k \neq n \end{cases}$$

$\Rightarrow$ If $n \neq m$ then $H_n(\mathbb{R}^n, \mathbb{R}^n-\{0\}) \not\approx H_n(\mathbb{R}^m, \mathbb{R}^m-\{0\})$.

Equivalence of simplicial and singular homology :

X : Δ-Complex $\imath : \Delta_n(X) \longrightarrow C_n(X)$ where $\imath$ maps each n-Simplex to its characteristic maps.

Prop $\imath_* : H_n^{\Delta}(X) \longrightarrow H_n(X)$ is an isomorphism.

Def $\underset{\text{subcomplex}}{A \subset X}$ $\Delta_n(X,A) = \dfrac{\Delta_n(X)}{\Delta_n(A)}$, $H_n^{\Delta}(X,A)$, satisfies a long exact seq. as in the singular homology.

proof : Suppose X is finite dim. We prove prop. by induction . Let $X^K \subset X$: Union of all simplicies of dim $\leqslant K$.

For $H_n^{\Delta}(X^0) \approx H_n(X^0)$. Suppose $\imath_* : H_n^{\Delta}(X^K) \longrightarrow H_n(X^K)$ is isom.

Then, consider (X^{K+1}, X^K) :

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}^{\Delta}(X^{K+1}, X^K) \arrow[r] \arrow[d] & H_n^{\Delta}(X^K) \arrow[r] \arrow[d, "\wr"] & H_n^{\Delta}(X^{K+1}) \arrow[r] \arrow[d] & H_n^{\Delta}(X^{K+1}, X^K) \arrow[r] \arrow[d] & H_{n-1}^{\Delta}(X^K) \arrow[r] \arrow[d, "\wr"] & \cdots \\
\arrow[r] & H_{n+1}(X^{K+1}, X^K) \arrow[r] & H_n(X^K) \arrow[r] & H_n(X^{K+1}) \arrow[r] & H_n(X^{K+1}, X^K) \arrow[r] & H_{n-1}(X^K) \arrow[r] & \cdots
\end{tikzcd}$$

Lem (Five Lemma) Suppose

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[r, "j"] \arrow[d, "\beta"'] & C \arrow[r, "K"] \arrow[d, "\boxed{\gamma}"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' \arrow[r, "k'"'] & D' \arrow[r, "l'"'] & E'
\end{tikzcd}$$

is a Comm. diag. of abelian group s.t. the first and second rows are exact. Then, if $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, δ is an isom, too.

Using five-lemma, we need to prove that the maps $H_n^{\Delta}(X^{K+1}, X^K) \longrightarrow H_n(X^{K+1}, X^K)$ are isomorphisms.

- $H_n^{\Delta}(X^{K+1}, X^K)$, $\Delta_n(X^{K+1}, X^K) = \dfrac{\Delta_n(X^{K+1})}{\Delta_n(X^K)} \Rightarrow \Delta_n(X^{K+1}, X^K) = 0$ if $n \neq K+1$

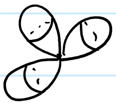
free abelian group gen. by (K+1)-Simplicies.

$\Rightarrow \quad \longrightarrow 0 \longrightarrow \Delta_{K+1}(X^{K+1}, X^K) \longrightarrow 0 \longrightarrow 0 \cdots$

$$\Rightarrow \quad H_n^{\Delta}(X^{K+1}, X^K) \approx \begin{cases} 0 & n \neq k+1 \\ \Delta_{K+1}(X^{K+1}, X^K) & n = K+1 \end{cases}$$

- $H_n(X^{K+1}, X^K)$ $\Phi : (\coprod_{\alpha} \Delta_{\alpha}^{K+1}, \coprod_{\alpha} \partial \Delta_{\alpha}^{K+1}) \longrightarrow (X^{K+1}, X^K)$

the induced map $\Phi : \coprod_{\alpha} \Delta_{\alpha}^{k+1} / \coprod_{\alpha} \partial \Delta_{\alpha}^{k+1} \longrightarrow X^{K+1}/X^K$ is a homeo.



$$\begin{tikzcd} H_n(\amalg_\alpha \Delta_\alpha^{k+1}, \amalg_\alpha \partial\Delta_\alpha^{k+1}) \arrow[r, "\Phi_*"] \arrow[d, "q_*"', "\wr"] & H_n(X^{k+1}, X^k) \arrow[d, "q_*"', "\wr"] \\ \tilde{H}_n(\amalg_\alpha \Delta_\alpha^{k+1} / \amalg_\alpha \partial\Delta_\alpha^{k+1}) \arrow[r, "\approx", "\Phi_*"'] & \tilde{H}_n(X^{k+1}/X^k) \end{tikzcd}$$

$\Rightarrow \Phi_*$ is an isom.

$$\Rightarrow H_n(X^{k+1}, X^k) \approx \begin{cases} 0 & n \neq k+1 \\ \downarrow & n = k+1 \end{cases}$$

free abelian group with a generator for each k+1 Simplex

$$H_{k+1}(\amalg_\alpha \Delta_\alpha^{k+1}, \amalg_\alpha \partial\Delta_\alpha^{k+1}) \approx \bigoplus_\alpha H_{k+1}(\Delta_\alpha^{k+1}, \partial\Delta_\alpha^{k+1})$$

<u>EX</u> For each K, $H_K(\Delta^K, \partial\Delta^K)$ is generated by the K-Simplex defined via $\mathbb{1} : \Delta^K \longrightarrow \Delta^K$.

$$\Rightarrow H_{K+1}^{\Delta}(X^{K+1}, X^K) \longrightarrow H_{K+1}(X^{K+1}, X^K) \text{ is an isom.}$$

For X infinite dim, use the fact that any compact subset $A \subset X$ intersects finitely many Simplicies.

<u>Rmk</u> For any subcomplex $A \subset X$, $H_n^{\Delta}(X, A) \approx H_n(X, A)$.