Lecture 17

Tuesday, March 28, 2017 9:59 PM

__Excision thm__ Given subspaces $Z \subset A \subset X$ s.t. $\overline{Z} \subset \mathrm{int}(A)$. Then

$$\imath_* : H_n(X-Z, A-Z) \longrightarrow H_n(X,A)$$

is an isom.

__Equivalently:__ Let $B = X - Z$, $\mathrm{int}\, B = X - \overline{Z} \rightsquigarrow \left(\overline{Z} \subset \mathrm{int}(A) \Leftrightarrow \mathrm{int}(A) \cup \mathrm{int}(B) = X \right)$

Given subspaces $A, B \subset X$ such that $\mathrm{int}(A) \cup \mathrm{int}(B) = X$. Then

$$\imath_* : H_n(B, B \cap A) \longrightarrow H_n(X, A)$$

induced by inclusion is an isom.

__Idea of proof__

$C_n(A+B)$: Subgroup of $C_n(X)$ which consists of the sum of n-chains in A and B

$$\cdots \longrightarrow C_n(A+B) \xrightarrow{\partial_n} C_{n-1}(A+B) \xrightarrow{\partial_{n-1}} C_{n-2}(A+B) \longrightarrow \cdots \quad : \text{chain Complex}$$

$\imath : C_*(A+B) \hookrightarrow C_*(X)$: gives a chain map (ρ)

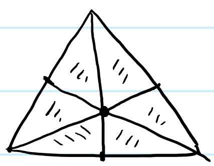
• $\imath$ is a chain homotopy equivalence. Construct a chain map ρ s.t. $\boxed{\rho\imath \simeq \mathbb{1}}$, $\imath\rho = \mathbb{1}$

$\partial D + D\partial = \rho\imath - \mathbb{1}$

ρ is Constructed by __barycentric__ subdivision.

v_0 b v_1

$[v_0, v_1] = [b, v_1] - [b, v_0]$



$\sigma : \Delta^n \longrightarrow X$

$\Delta^n = \Delta^n_1 + \cdots + \Delta^n_k \rightsquigarrow \sigma = \sum \sigma|_{\Delta^n_i}$

$S : C_n(X) \longrightarrow C_n(X)$

For each n-chain σ there exists $m \geq 0$ s.t. $S^m(\sigma) \in C_n(A+B)$

$\imath, \rho, D$ map $C_n(A)$ to itself $\Rightarrow$

$$\frac{C_n(B)}{C_n(A \cap B)} \hookrightarrow \frac{C_n(A+B)}{C_n(A)} \hookrightarrow \frac{C_n(X)}{C_n(A)}$$

isom

$\Rightarrow H_n(B, A \cap B) \approx H_n(X, A)$.

Cor For subspaces $A, B \subset X$ such that $\mathrm{int}(A) \cup \mathrm{int}(B) = X$ we get an exact seq. of the form

$$\cdots \xrightarrow{\partial} H_n(A\cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B) \longrightarrow \cdots$$

where:

$i : A\cap B \hookrightarrow A$

$j : A\cap B \hookrightarrow B$

$$\Rightarrow \Phi(\overset{H_n(A\cap B)}{\overset{\in}{x}}) = (i_*(x), -j_*(x))$$

$i' : A \hookrightarrow X$

$j' : B \hookrightarrow X$

$$\Psi(x, y) = i'_*(x) + j'_*(y)$$

Why we have such a long exact seq ?

Short exact seq: $0 \longrightarrow C_n(A\cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \longrightarrow 0$

$\varphi(x) = (x, -x)$ $\qquad \psi(x,y) = x+y$

Because: ① φ is injective

② $\ker(\psi) = \{(x,y) \mid x+y=0 \rightsquigarrow y=-x\} = \mathrm{Im}(\varphi)$

③ ψ is surjective

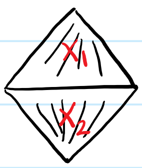
Since $C_n(A+B) \hookrightarrow C_n(X)$ induces an isom. on homology, we get the above long exact seq called Mayer-Vietoris sequence.

what is $\partial : H_n(X) \longrightarrow H_{n-1}(A\cap B)$? Let $\alpha \in H_n(X)$. Let $z \in \ker \partial_n$ be an n-cycle representing α. Use barycentric subdivision to write $z = x+y$ where $x \in C_n(A)$, $y \in C_n(B)$. $\partial z = \partial x + \partial y = 0 \Rightarrow \underline{\underline{\partial x}} = -\partial y = \beta$

$\Rightarrow \beta$ is an $(n-1)$-cycle in $A\cap B \Rightarrow \partial\alpha = [\beta]$.

Rmk $\cdots \longrightarrow \tilde H_n(A\cap B) \longrightarrow \tilde H_n(A) \oplus \tilde H_n(B) \longrightarrow \tilde H_n(X) \longrightarrow \tilde H_{n-1}(A\cap B) \longrightarrow \cdots$

Ex Let $Y = SX = \dfrac{X\times I}{X\times\{0\},\, X\times\{1\}} = X_1 \cup X_2$ $\quad$ Cone on X $\qquad X_1 \cap X_2 = X$



X

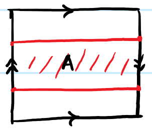
$$\cdots \longrightarrow \tilde H_n(X) \longrightarrow \underbrace{\cancel{\tilde H_n(X_1) \oplus \tilde H_n(X_2)}}_{0} \longrightarrow \tilde H_n(Y) \longrightarrow \tilde H_{n-1}(X) \longrightarrow \overbrace{\cancel{\tilde H_{n-1}(X_1) \oplus \tilde H_{n-1}(X_2)}}^{0} \longrightarrow \cdots$$

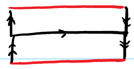
$$\Rightarrow \tilde H_n(Y) \approx \tilde H_{n-1}(X)$$

$X = S^{k-1} \rightsquigarrow Y = S^k \rightsquigarrow \tilde H_n(S^k) \approx \tilde H_{n-1}(S^{k-1})$ ✓

EX $X = Kb = A \cup B$

Möbius band





⟿ Möbius band

$$0 \to H_2(X) \xrightarrow{\partial_2} \overset{\mathbb{Z}}{\overset{\wr}{H_1(A\cap B)}} \xrightarrow{\Phi_1} \overset{\mathbb{Z}}{\overset{\wr}{H_1(A)}} \oplus \overset{\mathbb{Z}}{\overset{\wr}{H_1(B)}} \to H_1(X) \xrightarrow{\partial_2} \overset{\mathbb{Z}}{\overset{\wr}{H_0(A\cap B)}} \xrightarrow{\Phi_0} \overset{\mathbb{Z}}{\overset{\wr}{H_0(A)}} \oplus \overset{\mathbb{Z}}{\overset{\wr}{H_0(B)}} \to H_0(X) \to 0$$

$1 \longrightarrow (2,-2)$ $\qquad$ $1 \longrightarrow (1,-1)$ $\qquad$ $\Rightarrow H_0(X) \approx \mathbb{Z}$

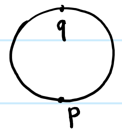
$\Rightarrow \Phi_1$ is injective. $\Rightarrow \partial = 0 \Rightarrow H_2(X) = 0$, Φ_0 : injective ⟿ $\ker \Phi_0 = 0 \Rightarrow \partial_2 = 0$

$$\Rightarrow \quad 0 \to \mathbb{Z} \xrightarrow{\Phi_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\Psi_1} H_1(X) \to 0$$

$1 \longrightarrow (2,-2)$ $\qquad$ surjective ⟿ $H_1(X) \approx \dfrac{\mathbb{Z}\oplus\mathbb{Z}}{\ker \Psi_1} = \dfrac{\overset{(1,-1),(1,0)}{\widetilde{\mathbb{Z}\oplus\mathbb{Z}}}}{\mathrm{Im}\,\Phi_1} \approx \mathbb{Z} \oplus \mathbb{Z}_2$

Ex $Y = S^1 \times X$ ⟿ Compute homology groups of Y in terms of homology groups of X.

$S^1 = U \cup V$, $U = S^1 \setminus \{p\}$, $V = S^1 \setminus \{q\}$ $\Rightarrow Y = U \times X \cup V \times X$



$$\to H_n((U\cap V)\times X) \xrightarrow{\Phi_n} H_n(U\times X) \oplus H_n(V\times X) \xrightarrow{\Psi_n} \boxed{H_n(Y)} \xrightarrow{\partial} H_{n-1}((U\cap V)\times X) \xrightarrow{\Phi_{n-1}}$$

$H_n((U\cap V)\times X) \approx H_n(X)\oplus H_n(X)$ $\qquad$ $(a,b) \longrightarrow (a+b, -a-b)$ $\qquad$ $H_{n-1}(U\times X) \oplus H_{n-1}(V\times X) \to \cdots$

$$0 \to \frac{H_n(X) \oplus \boxed{H_n(X)}}{\mathrm{Im}(\Phi_n)} \xrightarrow{\Psi_n} H_n(Y) \xrightarrow{\partial} \ker(\Phi_{n-1}) \to 0$$

$\mathrm{Im}(\Phi_n)$ generated by $(a,-a)$ for $a \in H_n(X)$ $\qquad$ $\ker(\Phi_{n-1})$ ↪ gen. by $(a,-a)$

$\Rightarrow$ Pick a basis for $H_n(X)$ ⟿ $(a,-a)$, $(0,a)$ give a basis for $H_n(X) \oplus H_n(X)$

$$0 \to H_n(X) \xrightarrow{\Psi_n} H_n(Y) \xrightarrow{\partial} H_{n-1}(X) \to 0 \quad ⟿ \quad H_n(Y) \overset{?}{\approx} H_n(X) \oplus H_{n-1}(X)$$

Special Case of short exact seq: $0 \to A \xrightarrow{i} \overset{B}{\overset{\|}{A\oplus C}} \xrightarrow{\pi} C \to 0$

Splitting lemma: Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a short exact seq. of abelian groups. Then the followings are equivalent:

(a) There exists $p: B \to A$ s.t. $pi = \mathbb{1}$

(b) " " $s: C \to B$ s.t. $js = \mathbb{1}$

(c) $B \approx A \oplus C$ such that

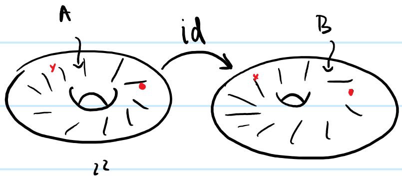
$$\begin{tikzcd} & B \arrow[dr] \arrow[d, phantom, "\wr"] & & \\ 0 \arrow[r] & A \arrow[ur] \arrow[dr] & C \arrow[r] & 0 \\ & A\oplus C \arrow[ur, "\pi"'] & & \end{tikzcd}$$

$\Rightarrow$ such a short exact seq. is called split.

In the example short exact seq. splits because, let $p_2 : Y = S^1 \times X \longrightarrow X$ be proj. then $(p_2)_* \circ \Upsilon_n = \mathbb{1}$

$$\Rightarrow \quad H_n(X \times S^1) \approx H_n(X) \oplus H_{n-1}(X).$$

Ex $H_n(S^1 \times S^1) \approx \begin{cases} \mathbb{Z} & n=2 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ \mathbb{Z} & n=0 \\ 0 & \text{otherwise} \end{cases}$ $\qquad H_k(S^n \times S^1) = \begin{cases} \mathbb{Z} & k=n+1 \\ \mathbb{Z} & k=n \\ \mathbb{Z} & k=1 \\ \mathbb{Z} & k=0 \\ 0 & \text{otherwise} \end{cases}$

Ex X: 

$A \cap B = T$

$\wr\wr$
$S^1 \times D^2$

$$0 \longrightarrow \tilde{H}_3(X) \longrightarrow \underset{\mathbb{Z}}{\tilde{H}_2(T)} \longrightarrow \underset{0}{\tilde{H}_2(A)} \oplus \underset{0}{\tilde{H}_2(B)} \longrightarrow \tilde{H}_2(X) \longrightarrow \tilde{H}_1(T) \xrightarrow{\Phi_1} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \longrightarrow \tilde{H}_1(X) \longrightarrow 0$$

$0 \longrightarrow \tilde{H}_3(X) \longrightarrow \mathbb{Z} \longrightarrow 0$

$\Downarrow$

$\tilde{H}_3(X) \approx \mathbb{Z}$

$0 \longrightarrow \tilde{H}_2(X) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\Phi_1} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \tilde{H}_1(X) \longrightarrow 0$

$(a,b) \longrightarrow (b,-b)$

$\tilde{H}_2(X) \approx \ker(\Phi_1) \approx \mathbb{Z}$

$\tilde{H}_1(X) \approx \mathbb{Z} \oplus \mathbb{Z} / \operatorname{Im} \Phi_1 \approx \mathbb{Z}$