Lecture 18

Monday, April 3, 2017 11:21 AM

Cellular homology

X: CW Complex

Goal: Compute homology of X from a chain complex $\{C_*^{CW}(X)\}$ where

$C_n^{CW}(X)$ has one basis element from each n-cell of X.

Def $C_n^{CW}(X) = H_n(X^n, X^{n-1})$ where X^n and X^{n-1} are n and $n-1$ skeletons of X $(X^{-1} = \emptyset)$

Note (X^n, X^{n-1}) : good pair $\rightsquigarrow$ $H_K(X^n, X^{n-1}) \approx \tilde{H}_K(\underbrace{X^n / X^{n-1}}_{\text{wedge sum of } n\text{-spheres}}) = \begin{cases} \text{free abelian} & K=n \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow H_n(X^n, X^{n-1})$: free abelian group with one generator corresponding to each n-cell.

Boundry map : $d_n = \partial_n^{CW} : C_n^{CW}(X) \longrightarrow C_{n-1}^{CW}(X)$ $\Rightarrow d_n = j_{n-1}\partial_n$

$$\begin{tikzcd}
C_n^{CW}(X) \arrow[r] \arrow[d, equal] & C_{n-1}^{CW}(X) \arrow[d, equal] \\
H_n(X^n, X^{n-1}) \arrow[r, dashed, "d_n"] \arrow[dr, "\partial_n"'] & H_{n-1}(X^{n-1}, X^{n-2}) \\
& H_{n-1}(X^{n-1}) \arrow[u, "j_{n-1}"']
\end{tikzcd}$$

Lem $d_n \circ d_{n+1} = 0$

$$\begin{tikzcd}
 & & H_{n-1}(X^{n-1}) \arrow[dr, "j_{n-1}"] & \\
H_{n+1}(X^{n+1}, X^n) \arrow[r, "d_{n+1}"] \arrow[dr, "\partial_{n+1}"'] & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[ur, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & H_n(X^n) \arrow[u, "j_n"'] & &
\end{tikzcd}$$

$d_n \circ d_{n+1} = j_{n-1} \circ \underbrace{\partial_n \circ j_n}_{=0} \circ \partial_{n+1} = 0$

$\Rightarrow$ **Def** $H_n^{CW}(X) = \dfrac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}$

Thm For any CW complex X, $H_n^{CW}(X) \approx H_n(X)$.

Lem ① $H_k(X^n) = 0$ for $k > n$. So if $\dim(X) = n$ then $H_k(X) = 0$ for $k > n$.

② If $k < n$, $H_k(X^n) \to H_k(X)$ induced by inclusion is an isom.

$k = n$ the map is surj.

Proof

$$\underset{\overset{\parallel}{0}}{H_{k+1}(X^{n+1}, X^n)} \to H_k(X^n) \to H_k(X^{n+1}) \to \underset{\overset{\parallel}{0}}{H_k(X^{n+1}, X^n)}$$

If $k \neq n, n+1 \Rightarrow H_k(X^n) \approx H_k(X^{n+1})$

$k = n \Rightarrow H_n(X^n) \to H_n(X^{n+1})$: Surj , $k = n+1 \Rightarrow H_{n+1}(X^n) \to H_{n+1}(X^{n+1})$: inj

$\Rightarrow$ **part 1** If $k > n$, $H_k(X^n) \approx H_k(X^{n-1}) \approx \cdots \approx H_k(X^0) = 0$

part 2 Suppose X is finite dim $\Rightarrow X = X^{n+m}$

$$H_k(X^n) \xrightarrow{\approx} H_k(X^{n+1}) \xrightarrow{\approx} \cdots \xrightarrow{\approx} H_k(X^{n+m}) \Rightarrow H_k(X^n) \approx \overset{\overset{H_k(X)}{\parallel}}{H_k(X^{n+m})} \text{ for } k < n$$

For $k = n \Rightarrow \underset{\text{surjective}}{H_n(X^n) \to H_n(X^{n+1})} \xrightarrow{\approx} H_n(X^{n+2}) \to \cdots \xrightarrow{\approx} H_n(X^{n+m}) \Rightarrow$ surjective

X: infinite dim, use the fact that each singular chain intersects finitely many cells of X.

proof of thm

$$\begin{tikzcd}
 & & H_{n-1}(X^{n-2}) = 0 \arrow[d] & \\
 & & H_{n-1}(X^{n-1}) \arrow[dr, "j_{n-1}\ \text{injective}"] & \\
H_{n+1}(X^{n+1}, X^n) \arrow[r, "d_{n+1}"] \arrow[dr, "\partial_{n+1}"'] & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[ur, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & H_n(X^n) \arrow[u, "j_n"'] \arrow[dr, "i_n"] & & \\
0 = H_n(X^{n-1}) \arrow[ur] & & H_n(X^{n+1}) \approx H_n(X) \arrow[d] & \\
 & & H_n(X^{n+1}, X^n) = 0 &
\end{tikzcd}$$

$\ker(d_n) = \ker(\partial_n) = \operatorname{im}(j_n) \Rightarrow j_n : \dfrac{H_n(X^n)}{\operatorname{Im}(\partial_{n+1})} \longrightarrow \dfrac{\ker(d_n) = \operatorname{im}(j_n)}{\operatorname{Im}(d_{n+1})}$

j_n : injective

$$H_n^{CW}(X) \approx \frac{H_n(X^n)}{\operatorname{Im}(\partial_{n+1})} = \frac{H_n(X^n)}{\ker(i_n)} \approx H_n(X). \ \blacksquare$$

Ex $X = \mathbb{C}P^k = e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2k}$

$$\Rightarrow\ 0 \xrightarrow{\overset{0}{\shortparallel}} \mathbb{Z} \xrightarrow{\overset{0}{\shortparallel}} 0 \xrightarrow{\overset{0}{\shortparallel}} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0$$

for any n $d_n = 0$

$$\Rightarrow\ H_n(\mathbb{C}P^k) = \begin{cases} \mathbb{Z} & \text{if } n = 2i \text{ for } 0 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

Ex $X = S^2 \vee S^4 \vee \cdots \vee S^{2k} = e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2k}$

(use formula for wedge sume)

$\Rightarrow H_n(X) \approx H_n(\mathbb{C}P^k)$ not homeomorphic !

Ex $X = \mathbb{R}P^k = e^0 \cup e^1 \cup \cdots \cup e^k$ $\quad C_n^{cw}(\mathbb{R}P^k) \cong \begin{cases} \mathbb{Z} & 0 \leq n \leq k \\ 0 & n > k \end{cases}$

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to 0$$

Compute boundry maps !

Degree :

lem For any homo $\varphi : \mathbb{Z} \to \mathbb{Z}$, there exists $d \in \mathbb{Z}$ s.t. $\varphi(\alpha) = d\alpha$.
($d = \varphi(1)$)

Def For a Conti map $f : S^n \to S^n$, $f_* : \underset{\mathbb{Z}}{\underset{\shortparallel}{\tilde{H}_n(S^n)}} \xrightarrow{\text{homo}} \underset{\mathbb{Z}}{\underset{\shortparallel}{\tilde{H}_n(S^n)}}$

$\Rightarrow$ degree of f : $\deg(f) = f_*(\overset{\overset{\mathbb{Z}}{\Downarrow}}{1})$

Properties
① $\deg \mathbb{1} = 1$
② $\deg fg = \deg f \deg g$ because $(fg)_* = f_* g_*$
③ $f \simeq g \Rightarrow \deg f = \deg g$
④ f: homotopy equiv. $\Rightarrow \deg f = \pm 1$
⑤ f is not surj $\Rightarrow \deg f = 0$

Cellular boundry map :

Let e^n_α be an n-cell and $\Phi_\alpha : (D^n_\alpha, \partial D^n_\alpha) \longrightarrow (X^n, X^{n-1})$ be it's charac. map. Then

choose a generator $[D^n_\alpha]$ for $H_n(D^n_\alpha, \partial D^n_\alpha) \cong \mathbb{Z}$, denote $\Phi_{\alpha *}([D^n_\alpha]) = [e^n_\alpha] \in H_n(X^n, X^{n-1})$

$[e^n_\alpha]$: generator for the $\mathbb{Z}$-summand in $H_n(X^n, X^{n-1})$ corresponding to e^n_α.

$\Rightarrow \{ [e^n_\alpha] : e^n_\alpha \text{ n-cell of } X \}$: basis for $C^{CW}_n(X)$

$d_n : C^{CW}_n(X) \longrightarrow C^{CW}_{n-1}(X)$

$$d_n([e^n_\alpha]) = \sum_\beta d_{\alpha\beta} [e^{n-1}_\beta] \quad \text{for some } d_{\alpha\beta} \in \mathbb{Z}$$

Lem $d_{\alpha\beta}$ is the deg of the map :

$$\partial D^n = S^{n-1} \xrightarrow[\text{attaching map}]{\varphi_\alpha} X^{n-1} \xrightarrow{q_\beta} X^{n-1}/X^{n-1} - e^{n-1}_\beta \cong S^{n-1}$$

$\varphi_\alpha = \Phi_\alpha |_{\partial D^n}$

Pf

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial", "\approx"'] \arrow[d, "\Phi_{\alpha *}"] & \tilde{H}_{n-1}(\partial D^n) \arrow[r, "\Delta_{\alpha\beta *} = (q_\beta \varphi_\alpha)_*"] \arrow[d, "\varphi_\alpha^*"] & \tilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "q"'] \arrow[ur, bend left, "q_\beta^*"] \arrow[d, "j_{n-1}"] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q'"'] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[ur, "\approx"'] &
\end{tikzcd}$$

$$d_n([e^n_\alpha]) = j_{n-1} \partial_n [e^n_\alpha] = j_{n-1} \partial_n \Phi_{\alpha *}[D_n] = j_{n-1} \overbrace{\partial \Phi_{\alpha *}[D_n]}^{\varphi_{\alpha *}[\partial D_n]}$$

$\rightsquigarrow$ in $\tilde{H}_{n-1}(X^{n-1}/X^{n-2})$ $\quad q_* \varphi_{\alpha *}[\partial D_n] \xrightarrow{\text{proj on the } \mathbb{Z}\text{-summand gen. by } [e^{n-1}_\beta]} q'_* q_* \varphi_{\alpha *}[\partial D_n] = q_{\beta *} \varphi_{\alpha *}[\partial D_n]$

$$= (\Delta_{\alpha\beta})_* [\partial D_n]$$

$\blacksquare$