Lecture 19

Tuesday, April 4, 2017 9:34 PM

Goal 1. Examples of Cellular homology / local degree
2. Euler Characteristic of a CW Complex

Recall $[e^n_\alpha]$: generator of the $\mathbb{Z}$ summand of $H_n(X^n, X^{n-1})$ corresponding to e^n_α

$$d_n([e^n_\alpha]) = \sum d_{\alpha\beta}[e^{n-1}_\beta]$$

$$d_{\alpha\beta} = \deg\Big(S^{n-1}_\alpha \longrightarrow X^{n-1} \longrightarrow \underbrace{X^{n-1}/X^{n-1} - e^{n-1}_\beta}_{S^{n-1}_\beta}\Big)$$

Degree: $f: S^n \to S^n$ then $f_*: \tilde{H}_n(S^n) \longrightarrow \tilde{H}_n(S^n)$, $\mathbb{Z} \longrightarrow \mathbb{Z}$ $\quad \deg f = f_*(1)$

Properties ① $\deg \mathbb{1} = 1$
② $\deg f \deg g = \deg fg$
③ $f \simeq g \rightsquigarrow \deg f = \deg g$

② and ③ $\Rightarrow$ If f is homotopy equivalence then $\deg f = \pm 1$

Ex $X = S^n \cup_\varphi e^{n+1}$ $\quad \varphi: S^n \to S^n$ of deg k

$$0 \to \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \to 0 \to \cdots \to \mathbb{Z} \to 0$$

d_n: multi by k

$\Rightarrow H_n(X) \approx \mathbb{Z}/d\mathbb{Z}$ $\quad H_i(X) = 0$ for $i \neq 0, n$ $\quad H_0(X) \approx \mathbb{Z}$

Remark $d_1: H_1(X^1, X^0) \longrightarrow H_0(X^0, \underbrace{X^{-1}}_{=\phi}) = H_0(X^0)$

$$H_1(X^1, X^0) \xrightarrow{\partial} H_0(X^0) \xrightarrow{j_0 = \mathbb{1}} H_0(X^0)$$

$\Rightarrow d_1 = \partial: H_1(X^1, X^0) \longrightarrow H_0(X^0)$ $\Rightarrow$ For a 1-cell e^1_α, $d_1[e^1_\alpha] = \partial^\Delta_1 e^1_\alpha$

$\alpha = [z] \longrightarrow [\partial z]$

relative 1-cycle $[z]$ $\Rightarrow \partial z$: 0-cycle in X^0

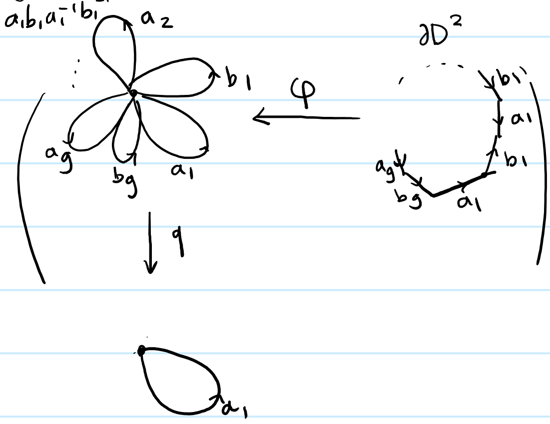
Simplicial boundry map

Ex $X = M_g$: 0-cell 1
1-cell $2g$
2-cell 1

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1 = 0} \mathbb{Z} \to 0$$

one 0-cell $\Rightarrow d_1 = 0$

$\underbrace{[a_1, b_1]}_{a_1 b_1 a_1^{-1} b_1^{-1}}[a_2, b_2] \cdots [a_g, b_g]$

$\deg \Big($  $\Big) = 0 \implies d_2 = 0 \implies H_n(M_g) \approx \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^{2g} & n = 1 \\ 0 & \text{otherwise} \end{cases}$

Ex (a) Compute homology groups of the nonorientable surface of genus g.

(b) Compute homology groups of an orientable surface of genus g with n boundary components.

Ex $X = \mathbb{R}P^K : e^0 \cup e^1 \cup \cdots \cup e^K \rightsquigarrow C_n^{CW}(X) \approx \mathbb{Z}$ for $0 \leq n \leq K$

$\rightsquigarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{d_K} \mathbb{Z} \xrightarrow{d_{K-1}} \cdots \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$

$X^{n+1} = \mathbb{R}P^{n+1} : \overset{X^n}{\overset{\|}{\mathbb{R}P^n}} \cup_\varphi e^{n+1}$

$\varphi : S^n \xrightarrow[\text{double covering map}]{} \mathbb{R}P^n = S^n / (x \sim -x)$

$d_n([e^{n+1}]) = \deg \left(\begin{array}{c} S^n = \partial D^{n+1} \xrightarrow{\varphi} \mathbb{R}P^n \\ \qquad\qquad\qquad \downarrow q \\ \qquad\qquad\qquad \mathbb{R}P^n / \mathbb{R}P^{n-1} \simeq S^n \end{array} \right) [e^n]$

what is the deg of $q\varphi$?!

Local degree

Assume $f : S^n \longrightarrow S^n$ is a Conti map, $x \in S^n$ and $y = f(x)$. Let $x \in U$ be an open nbd of x such that $f^{-1}(y) \cap U = \{x\}$. Then:

local homology group at x $\longrightarrow H_n(U, U-x) \overset{\text{excision}}{\approx} H_n(S^n, \underset{\sim\sim}{S^n - x}) \approx \tilde{H}_n(S^n) \approx \mathbb{Z}$

$f : (U, U-x) \longrightarrow (S^n, S^n - y) \rightsquigarrow f_* : H_n(U, U-x) \longrightarrow H_n(S^n, S^n - y)$

Let $[U]$ be a generator for $H_n(U, U-x)$ corresponding to generator $[S^n]$ for $\tilde{H}_n(S^n)$

$$\Rightarrow \quad f_*: H_n(U, U-x) \longrightarrow H_n(S^n, S^n-y)$$

$$f_*([U]) = \deg(f,x)[S^n]$$

$\deg(f,x)$: local degree of f at x

Properties ① $g, f: S^n \to S^n$ and $f(x) = y$, $g(y) = z$

$$\Rightarrow \deg(gf, x) = \deg(f, x)\deg(g, y)$$

② If $f: S^n \to S^n$ maps a nbd U of x homeo to a nbd V of y for $y = f(x)$, then $\deg(f,x) = \pm 1$.

$$f_*: H_n(U, U-x) \xrightarrow{\text{isom}} H_n(V, V-y) \approx H_n(S^n, S^n-y)$$

Thm Let $f: S^n \to S^n$ be a conti. map and $f^{-1}(y) = \{x_1, \dots, x_k\}$ for a $y \in S^n$.

Then $\deg(f) = \sum_{i=1}^{k} \deg(f, x_i)$.

pf Take disjoint open nbds $U_1, \dots, U_k$ s.t. $x_i \in U_i$.

$$\begin{tikzcd}
 & \underset{[S^n]}{\tilde{H}_n(S^n)} \arrow[r, "f_*"] \arrow[d, "j"] \arrow[dl, "\approx j"'] & \tilde{H}_n(S^n) \quad \deg(f)[S^n] \arrow[d, "j", "\approx"'] \\
H_n(S^n, S^n-x_i) & H_n(S^n, S^n-\{x_1, \dots, x_k\}) \arrow[r, "f_*"] \arrow[l, "P_i"'] & H_n(S^n, S^n-y) \quad \deg(f)[S^n]
\end{tikzcd}$$

$\tilde{H}_n(S^n-y) = 0$

$\tilde{H}_{n-1}(S^n-y) = 0$

Set $Z = S^n - U_1 \cup \dots \cup U_k$

$$H_n(S^n, S^n-\{x_1,\dots,x_k\}) \approx H_n\left(\bigcup_{i=1}^{k} U_i, \bigcup_{i=1}^{k} U_i - x_i\right) = \bigoplus_{i=1}^{n} H_n(U_i, U_i - x_i)$$

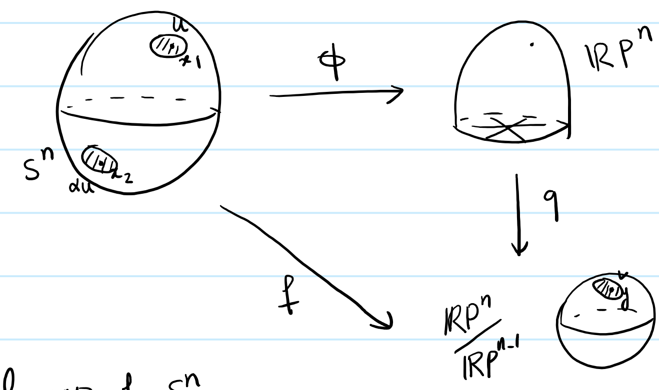
$$H_n(U_i, U_i - x_i) \approx H_n(S^n, S^n - x_i)$$

$\Rightarrow P_i j[S^n] = [S^n] \in H_n(S^n, S^n - x_i) \rightsquigarrow f_*\big|_{H_n(S^n, S^n-x_i)} [S^n] = \deg(f, x_i)[S^n]$

$$\Rightarrow f_* j[S^n] = \sum_{i=1}^{K} \deg(f, x_i)[S^n] \Rightarrow \deg f = \sum_{i=1}^{k} \deg(f, x_i)$$

Cor If f is not surjective, $\deg f = 0$

In $\mathbb{RP}^{n+1}$ example :



α : antipodal map of S^n

$\deg f = \deg(f, x_1) + \deg(f, x_2)$. Take a nbd u of x_1 such that f maps u homeo onto an open nbd of y.

$f\alpha = f$

$$\Rightarrow \deg(f, x_2) = \deg(f\alpha, x_2) = \deg(f, x_1) \underbrace{\deg(\alpha, x_2)}_{} \quad (\deg(\alpha, x_2) = \deg \alpha)$$

$$\Rightarrow \deg f = \deg(f, x_1) + \deg(f, x_2) = \deg(f, x_1)\left(1 + \deg \alpha\right) \qquad \alpha = -\mathbb{1}$$

Lem If $r_1 : S^n \longrightarrow S^n$ is reflection w.r. to $x_1 = 0$ hyperplane $\Rightarrow \deg r_1 = -1$.

i.e. $r_1(x_1, \dots, x_{n+1}) = (-x_1, x_2, \dots, x_{n+1})$

Pf S^n : Δ_1^n , Δ_2^n attach Δ_1 to Δ_2 with $id : \partial\Delta_1 \longrightarrow \partial\Delta_2$.

$\Rightarrow H_n(S^n)$: gen. by $\Delta_2^n - \Delta_1^n \xrightarrow{r_1} \Delta_1^n - \Delta_2^n \Rightarrow \deg r_1 = -1$.

Cor $\deg \alpha = (-1)^{n+1}$ $\quad \alpha(x_1, \dots, x_{n+1}) = r_{n+1} \cdots r_2 r_1 (x_1, \dots, x_n) \Rightarrow \deg(\alpha) = (-1)^{n+1}$

$\Rightarrow \deg(f) = \pm\left(1 + (-1)^{n+1}\right) \rightsquigarrow \deg(f) = 1 + (-1)^{n+1} \Rightarrow$

For $\mathbb{RP}^K$: $d_{n+1} = 1 + (-1)^{n+1} = \begin{cases} 0 & n+1 : \text{odd} \\ 2 & n+1 : \text{even} \end{cases}$ $\qquad \mathbb{Z} \xrightarrow[\ ]{d_{n+1} = 0} \mathbb{Z} \xrightarrow[\ ]{d_n = 2} \mathbb{Z}$

$$0 \to \mathbb{Z} \cdots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \Rightarrow H_n(\mathbb{RP}^K) \approx \begin{cases} \mathbb{Z} & n = 0, \text{ or } n = K \text{ and } K \text{ even} \\ \mathbb{Z}/2\mathbb{Z} & n : \text{odd}, \ 0 < n < K \\ 0 & \text{otherwise} \end{cases}$$

Euler characteristic

X: finite CW complex

Def : Euler charac. of X : $\chi(X) = c_0 - c_1 + c_2 - \cdots + (-1)^n c_n$ ↪ dim of X

c_i : # (i-cells)

Q Does $\chi(X)$ depend on the CW complex str ?

Thm $\chi(X) = \sum_{i=0}^{n} (-1)^i \, rk(H_i(X))$

$H_i(X)$: finitely gen. abelian group $\Rightarrow rk(H_i(X))$: # of $\mathbb{Z}$ summand in the decom. of $H_i(X)$ as a direct sum of cyclic groups.

Lem Given a short exact seq $0 \to A \to B \to C \to 0$ of finitely gen. abelian groups, then $rk(B) = rk(A) + rk(C)$

pf : $0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$

Let $Z_i = \ker d_i$

$B_i = \operatorname{Im} d_{i+1}$ $\Rightarrow 0 \to B_i \to Z_i \to \underset{\overset{\parallel}{Z_i/B_i}}{H_i(X)} \to 0$

$\Rightarrow rk(H_i) = rk(Z_i) - rk(B_i)$

$C_i \xrightarrow{d_i} C_{i-1} \Rightarrow 0 \to Z_i \hookrightarrow C_i \xrightarrow{d_i} B_{i-1} \to 0 \Rightarrow c_i = rk(B_{i-1}) + rk(Z_i)$

$$\sum_{i=0}^{n} (-1)^i c_i = \sum_{i=0}^{n} (-1)^i \left(rk(B_{i-1}) + rk(Z_i)\right) = \sum_{i=0}^{n} (-1)^i rk(B_{i-1}) + \sum_{i=0}^{n} (-1)^i rk(Z_i) = -\sum_{i=0}^{n} (-1)^i rk(B_i)$$

$$+ \sum_{i=0}^{n} (-1)^n rk(Z_i) = \sum_{i=0}^{n} (-1)^n rk(H_i) \checkmark$$