Lecture 2

Monday, January 23, 2017 12:09 PM

Recall

Prop If $A \subset X$ is a contractible subspace w/ HEP, then $q : X \longrightarrow X/A$ is a homotopy equivalence.

Def $A \subset X$. a homotopy relative A is a homotopy $f_t : X \longrightarrow Y$ s.t. $f_t|_A = f_0|_A$
i.e. indep. of t.

Ex $\underset{\text{subspace}}{A \subset X}$ and a deform. retrac. $r_t : X \longrightarrow X$ of X onto $A \Rightarrow r_t|_A = \mathbb{1}$
$\Rightarrow$ homotopy relative A.

Def $\underset{\text{subspace}}{A \subset X}$ and $\underset{\text{subspace}}{A \subset Y}$, then $f : X \longrightarrow Y$ is called a homotopy equiv. relative A

if $f|_A = \mathbb{1}$ and there exist $g : Y \longrightarrow X$ s.t. $g|_A = \mathbb{1}$ and $fg \simeq \mathbb{1}$ rel A
$gf \simeq \mathbb{1}$ rel A

$\Rightarrow$ we say $X \simeq Y$ rel A

Ex $\underset{\text{subspace}}{A \subset X}$ $r_t : X \longrightarrow X$ deforma. retrac. on to A s.t. $r_0 = r$, $r_1 = \mathbb{1}$ $\Rightarrow$ r is homotop. equ. relative A.

$$\begin{tikzcd} X \arrow[r, "r"] & X \arrow[l, bend left, "\mathbb{1}"] \end{tikzcd}$$

$r\mathbb{1} = r \simeq \mathbb{1}$ rel A
$\mathbb{1}r = r \simeq \mathbb{1}$ rel A

Prop : $\underset{\text{subspa.}}{A \subset X}$, $\underset{\text{subspace}}{A \subset Y}$, (X, A) and (Y, A) have HEP, Then any homotopy equi. $f : X \longrightarrow Y$

such that $f|_A = \mathbb{1}$ is a homotopy equivalence relative A.

- $r_t : X \longrightarrow X$ deform. retraction of X onto $A \subset X$ $\Longrightarrow$ $i : A \hookrightarrow X$ homotopy equivalence

  $\Longleftarrow$ If A has HEP

  $$\begin{tikzcd} A \arrow[r, hook, "i"] & X \arrow[l, dashed, bend left, "\bar{r}"] \end{tikzcd}$$

  $i\bar{r} = r_0 \simeq \mathbb{1}$
  $\bar{r}i = \mathbb{1}$

- $X = A$, $Y = X$ $\rightsquigarrow$ $i : A \hookrightarrow X$ homotopy equiv.
  $i|_A = \mathbb{1}$
  $\Rightarrow$ i is homotopy equiv. rel A.
  $\bar{r} : X \longrightarrow A$ s.t.
  $\bar{r}i = \mathbb{1}$
  $i\bar{r} \simeq \mathbb{1}$ rel A $\rightsquigarrow$ $r_t : X \longrightarrow X$
  homotopy between $\mathbb{1}$ and $i\bar{r}$ $\Rightarrow$ defor. retrac.

Cor $f: X \longrightarrow Y$ homotopy equiv. $\iff$ X is a defor. retraction of M_f

($X \simeq Y$ iff they are both deformation retractions of a third space)

Pf $X \subset M_f$ has HEP, $i: X \underset{\text{homotopy equivalence}}{\hookrightarrow} M_f$ $\iff$ f is homotopy equivalence

$f = ri$
$i \simeq jf$ $\Rightarrow$ ✓

Attaching Spaces:

X, Y: top spaces, $A \subset Y$ closed subspace
$f: A \longrightarrow X$

Attaching Y to X along A via f: $X \sqcup_f Y = \dfrac{X \sqcup Y}{(a \sim f(a) \mid \text{for all } a \in A)}$

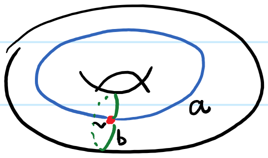
Ex $f: X \longrightarrow Y$ M_f: attaching $X \times I$ to Y along $X \times \{1\}$ via f.

Attaching an n-cell: X, D^n, $A = S^{n-1} = \partial D^n \subset D^n$, $\varphi: S^{n-1} \longrightarrow X$

$X \cup_\varphi D^n$: attaching an n-Cell to X via φ

CW Complexes: are formed inductively by attaching cells.

Ex

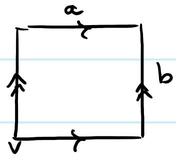


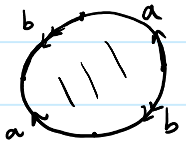
$X^0 = \{v\}$

$X^1 = (\underset{\underset{[0,1]}{\cong}}{D^1_a} \sqcup D^1_b) \cup_\varphi X^0$ $\varphi^1_a: \partial D^1_a \longrightarrow \{v\}$

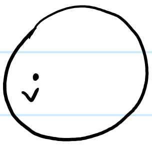
$\varphi^1_b: \partial D^1_b \longrightarrow \{v\}$

$X^2 = X^1 \cup_\varphi D^2$





Ex S^2



$X^0 = \{v\}$
$X^1 = X^0$
$X^2 = X^0 \cup_\varphi D^2$ $\varphi: S^1 \longrightarrow \{v\}$

CW-Complex: X^0: discrete set of pt, 0-Cell : 0-skeleton

$X^n = X^{n-1} \cup_\varphi (\sqcup_\alpha D^n_\alpha)$ $\varphi^n_\alpha: S^{n-1}_\alpha \longrightarrow X^{n-1}$

$e^n_\alpha := \operatorname{Int} D^n_\alpha \subset X$

- $X = \bigcup_{n\geqslant 0} X^n$ weak top. i.e $A \subset X$ is open (or closed) if $A \cap X^n$ is open (or closed) for each n.

<u>Characteristic map</u> : $\Phi_\alpha : D^n_\alpha \longrightarrow X$ homeom. on the interior

$$\hookrightarrow D^n_\alpha \hookrightarrow X^{n-1} \amalg D^n_\alpha \xrightarrow[\text{quotient}]{} X^n \hookrightarrow X$$

$$e^n_\alpha := \Phi_\alpha(\text{Int } D^n_\alpha)$$

<u>Ex</u> S^n : Cell complex with 2-cell $e^0 \cup e^n$

<u>Ex</u> $S^m \times S^n = e^0 \cup e^m \cup e^n \cup e^{m+n}$ (Special case: $m=n=1$)

X, Y Cell complexes $\rightsquigarrow$ $X \times Y$ is a cell complex with cells : $\underset{\text{Int } D^{n+m}}{\underset{\cong}{e^n_\alpha \times e^m_\beta}}$

<u>Ex</u> Real projective n-space $\mathbb{RP}^n = \{$1-dim subspaces of $\mathbb{R}^{n+1}$ i.e. all lines in $\mathbb{R}^{n+1}$ through origin$\}$

$$\cong \frac{S^n}{(v\sim -v \text{ for all } v\in S^n)} \cong \frac{D^n}{(p\sim -p \text{ for all } p\in S^{n-1})}$$

$$\cong \mathbb{RP}^{n-1} \cup_{\varphi_n} D^n \qquad \varphi_n : S^{n-1} \longrightarrow \mathbb{RP}^{n-1} = \frac{S^{n-1}}{p\sim -p}$$

$\Rightarrow \mathbb{RP}^n \cong e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n$

<u>Def</u> $RP^\infty = \bigcup_n \mathbb{RP}^n \cong e^0 \cup e^1 \cup \cdots \cup e^n \cup \cdots$

<u>Ex</u> Similarly, Complex proj. n-space $\mathbb{CP}^n$: Complex lines through origin in $\mathbb{C}^{n+1}$

$$\cong \mathbb{CP}^{n-1} \cup D^{2n} \cong e^0 \cup e^2 \cup \cdots \cup e^{2n}$$

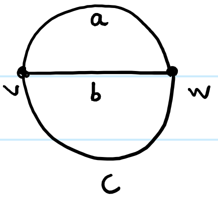
<u>Def</u> A <u>subcomplex</u> of X is a closed subspace $A \subseteq X$, s.t. it's a union of cells in X.

<u>Properties of CW complex</u>

- CW complexes are Hausdorff
- If $A \subset X$ is compact, then $A \subseteq$ finite subcomplex of X.
- If $A \subset X$ is a subcomplex, then it satisfies HEP.

<u>CoR</u> $A \subset X$ subcomplex contractible $\Rightarrow X/A \simeq X$

Ex $X =$ "Theta graph"



$A = a \cup \{v, w\}$

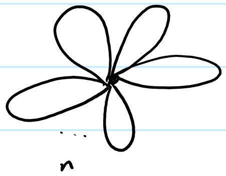
$\Rightarrow X \simeq X/A$

$S^1 \vee S^1$

In general, X: Connected graph with finitely many vertices and edges

$\Downarrow$

$$X \simeq \underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{n:\ \#\text{ of edges}}$$



Prop Assume $\underset{\text{closed}}{A \subset Y}$ and has HEP. If $f, g: A \to X$ s.t. $f \simeq g$

then $X \cup_f Y \simeq X \cup_g Y$.

Cor 2 To determine homotopy type of CW Complexes we only need attaching maps up to homotopy.

- Take $A = X^n \subset X$, homotopy type of X^{n+1} doesn't change attaching map up to homotopy.