Lecture 20

Monday, April 10, 2017 9:00 AM

- Euler characteristic
- Cohomology

X : topological space, Cohomology of X is constructed by dualizing the def of homology

G : abelian group

<u>Def</u> : Group of singular n-cochain w coefficients in G : $C^n(X;G) := \mathrm{Hom}(C_n(X), G)$

<u>Rmk</u> n-cochains $\Longleftrightarrow$ functions $\{\text{singular } n\text{-simplices}\} \longrightarrow G$

$\varphi : C_n(X) \longrightarrow G$

<u>Def</u> Coboundary map : $\delta^n : C^n(X;G) \longrightarrow C^{n+1}(X;G)$ $\delta = \partial^*$ i.e.

$$C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\varphi} G \rightsquigarrow \delta^n\varphi = \varphi\partial_{n+1} \in C^{n+1}(X;G)$$

$$(*) \quad 0 \longrightarrow C^0(X;G) \xrightarrow{\delta^0} C^1(X;G) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} C^n(X;G) \xrightarrow{\delta^n} C^{n+1}(X;G) \xrightarrow{\delta^{n+1}} \cdots$$

<u>Lem</u> $\delta^n\delta^{n-1} = 0$, Suppose $\varphi \in C^{n-1}(X;G)$ i.e.

$$C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\varphi} G$$

$$\delta^n\delta^{n-1}\varphi = \delta^n\varphi\partial_n = \varphi\,\underbrace{\partial_n\partial_{n+1}}_{=0} = 0$$

<u>Def</u> $H^n(X;G) = \dfrac{\ker \delta^n}{\mathrm{Im}\, \delta^{n-1}}$ Elements of $\ker\delta$: cocycle

" " $\mathrm{Im}\,\delta$: coboundary

<u>Lem</u> $H^0(X;G) = \mathrm{Hom}(H_0(X);G)$

<u>Pf</u>

$H^0(X;G) = \ker \delta^0$ $\quad C^0(X;G) \longrightarrow C^1(X,G)$

$\Updownarrow$

pts in $X \Longleftrightarrow$ functions on singular 0-simplices

Any $\varphi \in C^0(X;G)$ is a $\varphi : X \longrightarrow G$, $\delta\varphi = \varphi\partial = 0 \Longleftrightarrow \varphi\partial\sigma = 0$ for any $\sigma : [v_0, v_1] \longrightarrow X$

$\Longleftrightarrow \varphi(\sigma(v_1) - \sigma(v_0)) = \varphi(\sigma(v_1)) - \varphi(\sigma(v_0)) = 0 \rightsquigarrow \varphi$ is constant on any path connected component of X

Relative Cohomology: $A \underset{\text{subspace}}{\subset} X$ $\quad C^n(X, A; G) = \mathrm{Hom}(C_n(X,A), G)$ $\ni \varphi$

Short exact seq: $0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \to 0$

Lem: $0 \leftarrow C^n(A;G) \xleftarrow{i^*} C^n(X;G) \xleftarrow{j^*} C^n(X,A;G) \leftarrow 0$ is exact. $(*)$

Pf ① i^* is surjective.

$\varphi \in C^n(X;G) \Rightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{\varphi} G$ $\quad i^*\varphi = \varphi i \Rightarrow i^*$: restriction of φ to singular n-simplicies in A.

Given a function from singular n-simplicies in A, we may extend it all n-simplices in X by for example defining it equal 0 on singular n-simplicies that aren't in A.

② $\underbrace{\mathrm{Ker}\, i^*}$ $= \mathrm{im}\, j^*$

Cochains that are 0 on singular n-simplicies in A.

$$\varphi \in \mathrm{Ker}\, i^* \Rightarrow \varphi : \frac{C_n(X)}{C_n(A)} \longrightarrow G \Rightarrow \varphi \in \mathrm{im}(j^*)$$

$\frac{C_n(X)}{C_n(A)} = C_n(X,A)$

③ j^*: injective $\quad C_n(X) \xrightarrow{j} C_n(X,A) \xrightarrow{\varphi} G \quad j^*\varphi = \varphi j \neq 0$

$C_n(X,A) = C_n(X)/C_n(A)$

Rmk Any $\varphi \in C^n(X,A;G)$: function from Singular n-simplicies in X to G which vanishes on singular n-simplicies in A.

Def $\delta^n : C^n(X;G) \longrightarrow C^{n+1}(X;G) \rightsquigarrow \delta^n\big|_{C^n(X,A;G)} : C^n(X,A;G) \longrightarrow C^{n+1}(X,A;G)$

$(*)$ is a short exact seq. of cochain complexes.

$\Rightarrow$ Long exact seq:

$$\cdots \to H^{n-1}(A;G) \xrightarrow{\delta} H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \longrightarrow \cdots$$

Induced homo: $f: X \longrightarrow Y \Rightarrow f_\# : C_n(X) \longrightarrow C_n(Y) \quad f_\# \partial = \partial f_\#$

$\Rightarrow f^\# : C^n(Y;G) \longrightarrow C^n(X;G) \quad \underline{\delta f^\# = f^\# \delta}$ cochain map

$\Rightarrow f^* : H^n(Y;G) \longrightarrow H^n(X;G)$

properties ① $\mathbb{1}^* = \mathbb{1}$

② $(fg)^* = g^* f^*$

③ $f, g : X \longrightarrow Y$, $f \simeq g \Rightarrow f^* = g^*$

④ $f:(X,A)\longrightarrow(Y,B) \rightsquigarrow f^*: H^n(Y,B;G)\longrightarrow H^n(X,A;G)$

Diagram Commutes :

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(Y,B;G) \arrow[r, "j^*"] \arrow[d, "f^*"] & H^n(Y;G) \arrow[r, "i^*"] \arrow[d, "f^*"] & H^n(B;G) \arrow[r, "\delta"] \arrow[d, "f^*"] & H^{n+1}(Y,B;G) \arrow[r] \arrow[d, "f^*"] & \cdots \\
\cdots \arrow[r] & H^n(X,A;G) \arrow[r, "j^*"] & H^n(X;G) \arrow[r, "i^*"] & H^n(A;G) \arrow[r, "\delta"] & H^{n+1}(X,A;G) \arrow[r] & \cdots
\end{tikzcd}$$

If $f\simeq g:(X,A)\longrightarrow(Y,B) \Rightarrow f^*=g^*$.

<u>Excision</u> : $Z\subset A\subset X$ s.t. $\overline{Z}\subset \operatorname{int}(A)$

$\Rightarrow i:(X-Z,A-Z)\hookrightarrow(X,A) \dashrightarrow i^*: H^n(X,A;G)\xrightarrow{\approx} H^n(X-Z,A-Z;G)$

<u>Mayer-Vietoris sequence</u> : $X=\operatorname{int}(A)\cup\operatorname{int}(B)$

$$\cdots\longrightarrow H^n(X;G)\xrightarrow{\Psi} H^n(A;G)\oplus H^n(B;G)\xrightarrow{\Phi} H^n(A\cap B;G)\longrightarrow H^{n+1}(X;G)\longrightarrow\cdots$$

<u>Simplicial cohomology</u> : X: Δ-Complex $\rightsquigarrow \Delta^n(X;G)=\operatorname{Hom}(\Delta_n(X);G)$, $\delta=\partial^*$

$H^n_\Delta(X;G)\approx H^n(X,G)$.

<u>Cellular cohomology</u> : X: CW-Complex $\quad X^n$: n-skeleton

cellular cochain complex :

$$\begin{tikzcd}
 & & H^{n-1}(X^{n-1};G) \arrow[dr, "\delta_{n-1}"] & & & & \\
\cdots \arrow[r] & H^{n-1}(X^{n-1},X^{n-2};G) \arrow[ur, "j_{n-1}"] \arrow[rr, "d_{n-1}"', "\underset{\delta_{n-1}\circ j_{n-1}}{\shortparallel}"] & & H^n(X^n,X^{n-1};G) \arrow[dr, "j_n"] \arrow[rr, "d_n"] & & H^{n+1}(X^{n+1},X^n;G) \arrow[r] & \cdots \\
 & & & & H^n(X^n;G) \arrow[ur, "\delta_n"] & &
\end{tikzcd}$$

<u>Thm</u> ① $H^n_{CW}(X;G):=\dfrac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n-1}}\approx H^n(X;G)$

② Cellular cochain Complex is isomorphic to dual of the cellular chain Complex by applying $\operatorname{Hom}(-,G)$.

<u>EX</u> $S^n=e_0\cup e_n$

$\rightsquigarrow \quad 0\longrightarrow \mathbb{Z}\xrightarrow{0} 0\xrightarrow{0}\cdots\xrightarrow{0}\mathbb{Z}\longrightarrow 0$

$\quad\;\; 0\longleftarrow G\xleftarrow{0} 0\xleftarrow{0}\cdots\xleftarrow{0} G\longleftarrow 0 \rightsquigarrow H^k(S^n;G)\approx\begin{cases} G & k=0,n \\ 0 & \text{otherwise}\end{cases}$

Ex $X = \mathbb{R}P^k = e^0 \cup e^1 \cup \cdots \cup e^n \qquad G = \mathbb{Z}$

$0 \longrightarrow \mathbb{Z} \xrightarrow{0 \text{ or } 2} \quad \cdots \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$: cellular chain complex

$0 \longleftarrow \mathbb{Z} \longleftarrow \quad \cdots \mathbb{Z} \xleftarrow{0} \underline{\underline{\mathbb{Z}}} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow 0$ cellular cochain complex

k : odd

k : even $H^n(\mathbb{R}P^k; \mathbb{Z}) \approx \begin{cases} \mathbb{Z} & n = k, 0 \\ \mathbb{Z}/2\mathbb{Z} & n: \text{even and } \le k \\ 0 & \text{otherwise} \end{cases}$

$H^k(\mathbb{R}P^k; \mathbb{Z}) \approx \mathbb{Z}/2\mathbb{Z}$

- $H^n(\mathbb{R}P^k; \mathbb{Z})$ is <u>not</u> $\mathrm{Hom}(H_n(\mathbb{R}P^k); \mathbb{Z})$.