Lecture 21

Tuesday, April 11, 2017 9:21 PM

① Examples

② Universal coefficient thm : describes $H^n(X;G)$ in terms of homology groups of X and the abelian group G.

Ring : R

__Def__ Free resolution of an abelian group H (R-module M) is an exact sequence (i.e. $\ker f_n = \operatorname{Im} f_{n+1}$)

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0 \quad (M)$$

where each F_i is a free abelian group (R-module).

Take a free resolution for H and dualize it :

$$\cdots \xleftarrow{f_3^*} F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \leftarrow 0$$

denote $H^n(F;G) = \dfrac{\ker f_{n+1}^*}{\operatorname{Im} f_n^*}$

__Lem__ For any two free resolutions F and F' of H, $H^n(F;G) \approx H^n(F';G)$ for all n.

__Def__ $\operatorname{Ext}^n(M,G) := H^n(F;G)$.

Any abelian group H has a free resolution of the form $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$

pick a set of generators for H and let F_0 be a free abelian group generated by a basis in one-to-one correspondance with these generator.

$F_0 \xrightarrow{f_0} H$ the surjective map defined by the one-to-one correspondance.

$F_1 = \ker(f_0)$: free

__Ex__ $H = \mathbb{Z}_n$ $\quad 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$

__Cor__ $H^n(F;G) = 0$ for $n \geq 2$.

$$0 \leftarrow F_1 \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \leftarrow 0$$

__Ex__ f_0^* is injective , $\operatorname{im}(f_0^*) = \ker f_1^* \rightsquigarrow H^0(F;G) = 0$

__Def__ $\operatorname{Ext}(H,G) := H^1(F;G)$ $\quad$ free resolution for H.

__Properties__ ① $\operatorname{Ext}(H \oplus H', G) \approx \operatorname{Ext}(H,G) \oplus \operatorname{Ext}(H',G)$

② H : free $\Rightarrow \operatorname{Ext}(H,G) = 0$ $\quad 0 \to H \to H \to 0$

③ $\mathrm{Ext}(\mathbb{Z}_n, G) \approx G/nG$ $\qquad 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$

$$0 \leftarrow \boxed{G \xleftarrow{n} G} \leftarrow \mathrm{Hom}(\mathbb{Z}_n, G) \leftarrow 0$$

Given a finitely gen. abelian group H: $\quad H = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \underbrace{\mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n}}_{\text{torsion subgroups of } H}$

$$\mathrm{Ext}(H, G) \approx G/d_1G \oplus G/d_2G \oplus \cdots \oplus G/d_nG$$

<u>Thm</u> (Universal coefficient thm) C: chain complex of free abelian groups

$$H^n(C; G) \approx \mathrm{Hom}(H_n(C), G) \oplus \mathrm{Ext}(H_{n-1}(C), G)$$

Cohomology groups of the dual cochain complex $C^n = \mathrm{Hom}(C_n; G)$

<u>Special case</u> $G = \mathbb{Z}$. If homology groups $H_n(C)$ and $H_{n-1}(C)$ are finitely generated then

$$H^n(C; \mathbb{Z}) \approx (H_n/T_n) \oplus T_{n-1}$$

T_{n-1}: torsion subgroups of $H_{n-1}(C)$

T_n: torsion subgroups of $H_n(C)$.

<u>proof</u> Construct an split exact sequence:

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to \underset{\frac{\ker(\delta^n)}{\mathrm{Im}(\delta^{n+1})}}{\overset{\shortparallel}{H^n(C; G)}} \xrightarrow{h} \mathrm{Hom}(H_n(C); G) \to 0$$

• Construct h:

Suppose $\underset{C_n \to G}{\varphi} \in \ker \delta^n \Rightarrow \varphi\partial = 0 \Rightarrow \varphi$ vanishes on $B_n = \mathrm{im}\,\partial$.

$$\varphi: C_n \to G \Rightarrow \varphi_0 = \varphi|_{\underset{\ker\partial}{Z_n}} : Z_n \to G \rightsquigarrow \overline{\varphi}_0 : \overset{H_n(C)}{\overset{\shortparallel}{Z_n/B_n}} \to G$$

If $\varphi = \delta\psi = \psi\partial \rightsquigarrow \varphi|_{Z_n} = 0 \quad \Rightarrow \quad H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G)$

$$[\varphi] \longmapsto \overline{\varphi}_0$$

• h is surjective: $\overline{\varphi}_0 : Z_n/B_n \to G \rightsquigarrow \varphi_0 = \overline{\varphi}_0 \circ q : Z_n \to G$

$(q: Z_n \to Z_n/B_n) \qquad \varphi_0|_{B_n} = 0$

Short exact seq: $0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial} \overset{\text{free}}{\boxed{B_{n-1}}} \to 0 \Leftarrow$ Exact sequence splits.

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$\Rightarrow$ There exists a proj. $p: C_n \to Z_n$ s.t. $p i_n = \mathbb{1} \rightsquigarrow$ let $\varphi = \varphi_0 p : C_n \to G$.

$\varphi|_{Z_n} = \varphi_0 \rightsquigarrow \varphi|_{B_n} = 0 \Rightarrow \delta\varphi = \varphi\partial = 0 \Rightarrow \varphi \in \ker\delta \Rightarrow [\varphi] \in H^n(C; G)$. $h[\varphi] = \overline{\varphi}_0$

$\Rightarrow$ Split short exact seq: $0 \to \ker h \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C),G) \to 0$

• $\ker h \approx \mathrm{Ext}(H_{n-1}(C), G)$

Follows from a long exact sequ:

$$\cdots \leftarrow B_n^* \xleftarrow{i_n^*} Z_n^* \leftarrow H^n(C;G) \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \cdots \qquad \text{(page 192)}$$

$$\Downarrow$$

Short exact seq $\quad 0 \leftarrow \ker(i_n^*) \xleftarrow{h} H^n(C;G) \leftarrow \mathrm{Coker}(i_{n-1}^*) = \dfrac{B_{n-1}^*}{\mathrm{Im}\, i_{n-1}^*} \leftarrow 0$

$$\ker(i_n^*) = \{\varphi : Z_n \to G \mid \varphi|_{B_n} = 0\} \Rightarrow \varphi : Z_n/B_n \to G \Rightarrow \varphi : H_n(C) \to G$$
$$\Rightarrow \varphi \in \mathrm{Hom}(H_n(C), G)$$

$\mathrm{Coker}(i_{n-1}^*)$:

$$0 \to \underbrace{B_{n-1}}_{} \xrightarrow{i_{n-1}} \underbrace{Z_{n-1}}_{} \to H_{n-1}(C) \to 0 \;:\; \text{Free resolution of } H_{n-1}(C)$$

free group

$$\Rightarrow \quad 0 \leftarrow \boxed{B_{n-1}^*} \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \mathrm{Hom}(H_{n-1}(C), G) \leftarrow 0$$

$$\mathrm{Ext}(H_{n-1}(C), G) = \frac{B_{n-1}^*}{\mathrm{Im}(i_{n-1}^*)} = \mathrm{Coker}(i_{n-1}^*)$$

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Ex $\quad X = \mathbb{RP}^3 \qquad 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \qquad H_n(\mathbb{RP}^3) \approx \begin{cases} \mathbb{Z} & n = 0, 3 \\ \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$

$$H^n(\mathbb{RP}^3; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 3, 0 \\ 0 & n = 1 \\ \mathbb{Z}_2 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Rmk $\quad (X, A) \rightsquigarrow \quad \cdots \to H^{n-1}(A,G) \xrightarrow{\delta} H^n(X,A;G) \to H^n(X;G) \to H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \cdots$

Diagram is Commutative:

$$\begin{array}{ccc} H^n(A;G) & \xrightarrow{\delta} & H^{n+1}(X,A;G) \\ \downarrow h & & \downarrow h \\ \mathrm{Hom}(H_n(A), G) & \xrightarrow{\partial^*} & \mathrm{Hom}(H_{n+1}(X,A), G) \end{array}$$