Lecture 22

Sunday, April 16, 2017 4:45 PM

Künneth formula: X and Y are top spaces, Compute $H_*(X\times Y)$ and $H^*(X\times Y;G)$ in terms of $H_*(X), H_*(Y)$ and $H^*(X;G), H^*(Y;G)$.

Def A, B are abelian groups (R-modules). Tensor product of A and B is an abelian group (R-module) defined as

$$\underset{A\otimes_R B}{A\otimes B} := \frac{\overset{\text{R-module}}{\text{Free abelian group w. basis } \{a\otimes b \mid a\in A, b\in B\}}}{\left\langle \begin{array}{l} a_1\otimes b + a_2\otimes b = (a_1+a_2)\otimes b \\ a\otimes b_1 + a\otimes b_2 = a\otimes(b_1+b_2) \end{array}\right\rangle + ra\otimes b = a\otimes rb}$$

Ex. $\mathbb{Z}\otimes\mathbb{Z} \approx \mathbb{Z}$ $\quad n\otimes m = 1\otimes m + 1\otimes m + \cdots + 1\otimes m = 1\otimes nm$, $\quad n\otimes m \overset{\mathbb{Z}}{\mapsto} nm$

- For any A $\quad \mathbb{Z}\otimes A \approx A$
- $(\bigoplus A_i)\otimes B \approx \bigoplus_i (A_i\otimes B)$

Cor: A: free abelian group w. basis $\{a_i\}_I$
B: " " " " " $\{b_j\}_J$ $\rightsquigarrow$ $A\otimes B$: free abelian group with basis $\{a_i\otimes b_j\}_{i\in I, j\in J}$

Ex $A\otimes\mathbb{Z}_n \approx A/nA$ for any A.

- Any bilinear map $\varphi: A\times B \to C$ induces a homo $\varphi: A\otimes B \to C$
$\varphi(a\otimes b) = \varphi(a,b)$.

Def Let C and C' be chain complexes of abelian groups. Then the tensor product chain complex $C\otimes C'$ is defined as:

$$C\otimes C' : \quad \cdots \longrightarrow (C\otimes C')_n \xrightarrow{\partial} (C\otimes C')_{n-1} \longrightarrow \cdots$$

where $(C\otimes C')_n = \bigoplus_{i=0}^{n} (C_i\otimes C'_{n-i})$: free abelian

$$\partial(\underset{C_i}{c}\otimes \underset{C'_{n-i}}{c'}) = (\partial c)\otimes c' + (-1)^i c\otimes(\partial c')$$

It's a chain complex because: $\partial^2(c\otimes c') = \partial(\underset{C_{i-1}}{\partial c}\otimes c' + (-1)^i c\otimes \partial c')$

$$= \underbrace{(\partial^2 c)}_{0}\otimes c' + (-1)^{i-1}\partial c\otimes\partial c' + (-1)^i\partial c\otimes\partial c' + (-1)^i c\otimes\underbrace{\partial^2 c'}_{0} = 0$$

Thm (Eilenberg-Zilber) For top spaces X and Y

Singular chain Complex $\rightsquigarrow$ $C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y)$

chain homotopy equivalent.

Suppose X and Y are CW Complexes $\Rightarrow$ $X \times Y$ is a CW complex

n-cell in $X \times Y$ $\Longleftrightarrow$ $e^i \times e^{n-i}$ where e^i : i-cell in X

e^{n-i} : $(n-i)$-cell in Y

$\Rightarrow$ $C_n^{CW}(X \times Y) \approx \bigoplus_i C_i^{CW}(X) \otimes C_{n-i}^{CW}(Y)$.

Prop 3B.1 Under this isom $d(e^i \otimes e^{n-i}) = de^i \otimes e^{n-i} + (-1)^i e^i \otimes de^{n-i}$

Q How to compute homology groups of $C \otimes C'$ interms of homology groups of C and C'?

Obs $\partial(c \otimes c') = \partial c \otimes c' + (-1)^i c \otimes \partial c'$ $\quad c \in C_i$

- $\partial c = \partial c' = 0 \Rightarrow \partial(c \otimes c') = 0$ $\qquad Z_i \otimes Z'_{n-i} \subset \ker(\partial)$
- $\partial c' = 0 \quad c = \partial \tilde{c} \rightarrow \partial(\tilde{c} \otimes c') = c \otimes c' \rightsquigarrow B_i \otimes Z'_{n-i} \subset \operatorname{Im}(\partial)$

$Z_i \otimes B'_{n-i} \subset$

$\Rightarrow H_i(C) \otimes H_{n-i}(C') \xrightarrow{\times} H_n(C \otimes C')$

Thm If C is a chain complex of free abelian groups then we have a split exact seq:

$$0 \longrightarrow \bigoplus_i H_i(C) \otimes H_{n-i}(C') \longrightarrow H_n(C \otimes C') \longrightarrow \bigoplus_i \operatorname{Tor}(H_i(C), H_{n-i-1}(C')) \longrightarrow 0$$

Def Let A and B be abelian groups. Take a free resolution $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \longrightarrow 0$ for A. Then

$$0 \longrightarrow \operatorname{Tor}(A,B) = \ker(f_1 \otimes \mathbb{1}) \longrightarrow F_1 \otimes B \xrightarrow{f_1 \otimes \mathbb{1}} F_0 \otimes B \xrightarrow{f_0 \otimes \mathbb{1}} A \otimes B \longrightarrow 0$$

exact

Properties
① $\operatorname{Tor}(A,B) \approx \operatorname{Tor}(B,A)$
② $\operatorname{Tor}(\bigoplus_i A_i, B) \approx \bigoplus_i \operatorname{Tor}(A_i, B)$
③ If A or B is free, then $\operatorname{Tor}(A,B) = 0$ $\qquad 0 \to A \to A \to 0 \rightsquigarrow 0 \to A \otimes B \to A \otimes B \to 0$
④ $\operatorname{Tor}(\mathbb{Z}_n, B) = \ker(B \xrightarrow{n} B)$

$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}_n \to 0 \Rightarrow 0 \to \operatorname{Tor}(\mathbb{Z}_n, B) \longrightarrow B \xrightarrow{n} B \longrightarrow B/nB \to 0$

Prop A is a free abelian group $\Leftrightarrow \mathrm{Tor}(A,B)=0$ for all B.

Cor $A = \mathbb{Z}\oplus\cdots\oplus\mathbb{Z}\oplus\mathbb{Z}_{d_1}\oplus\cdots\oplus\mathbb{Z}_{d_m}$

$\rightsquigarrow \mathrm{Tor}(A,B)\approx \ker(B\xrightarrow{d_1}B)\oplus\cdots\oplus\ker(B\xrightarrow{d_m}B)$

Ex $\mathrm{Tor}(\mathbb{Z}_n,\mathbb{Z}_m) = \ker(\mathbb{Z}_m\xrightarrow{n}\mathbb{Z}_m)\approx\mathbb{Z}_d$ for $d=\gcd(n,m)$

Thm (Künneth formula) $H_n(X\times Y)\approx \bigoplus_i (H_i(X)\otimes H_{n-i}(Y))\oplus\bigoplus_i \mathrm{Tor}(H_i(X),H_{n-i-1}(Y))$

Def $H_i(X)\otimes H_{n-i}(Y)\longrightarrow H_n(X\times Y)$: Cross product map

Cor If $H_i(X)$ is a free group for all i, then $H_n(X\times Y)\approx\bigoplus_i(H_i(X)\otimes H_{n-i}(Y))$

Ex $X\times S^K$ $\quad H_i(S^K)\approx\begin{cases}\mathbb{Z} & i=0,K\\ 0 & \text{otherwise}\end{cases}\quad\Rightarrow H_n(X\times S^K)\approx H_n(X)\oplus H_{n-K}(X)$

Ex $X=\mathbb{T}^3=S^1\times S^1\times S^1=T^2\times S^1 \rightsquigarrow H_3(\mathbb{T}^3)\approx H_2(T)\approx\mathbb{Z}$

$H_2(\mathbb{T}^3)\approx H_2(T)\oplus H_1(T)\approx\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$

$H_1(\mathbb{T}^3)\approx H_1(T)\oplus H_0(T)\approx\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$

$H_0(\mathbb{T}^3)\approx\mathbb{Z}$

Ex $X=S^1\cup_\varphi e^2\quad \deg(\varphi)=m\quad\Rightarrow 0\to\mathbb{Z}\xrightarrow{m}\mathbb{Z}\xrightarrow{0}\mathbb{Z}\to 0\qquad H_i(X)\approx\begin{cases}\mathbb{Z} & i=0\\ \mathbb{Z}_m & i=1\\ 0 & \text{otherwise}\end{cases}$

$Y=S^1\cup_\psi e^2\quad\deg(\psi)=n\quad\Rightarrow H_i(Y)\approx\begin{cases}\mathbb{Z} & i=0\\ \mathbb{Z}_n & i=1\\ 0 & \text{otherwise}\end{cases}$

$H_1(X\times Y)\approx H_1(X)\otimes H_0(Y)\oplus H_0(X)\otimes H_1(Y)\oplus \underbrace{\mathrm{Tor}(H_0(X),H_0(Y))}_{=0}\approx\mathbb{Z}_m\oplus\mathbb{Z}_n$

$H_2(X\times Y)\approx H_1(X)\otimes H_1(Y)\oplus\underbrace{\mathrm{Tor}(H_0(X),H_1(Y))}_{=0}\oplus\underbrace{\mathrm{Tor}(H_1(X),H_0(Y))}_{=0}\approx\mathbb{Z}_d$

$d=\gcd(m,n)$

$H_3(X\times Y)\approx\mathrm{Tor}(H_1(X),H_1(Y))\approx\mathbb{Z}_d$

$H_n(X\times Y)=0$ for $n>3$.

Cor (Universal Coeff thm for homology)

X: top space
G: abelian group $\rightsquigarrow\quad\cdots\longrightarrow C_n(X)\otimes G\xrightarrow{\partial\otimes\mathbb{1}}C_{n-1}(X)\otimes G\longrightarrow\cdots$

$\partial\otimes\mathbb{1}(c\otimes g)=\partial c\otimes g$

$\Rightarrow\quad H_n(X;G):=H_n(C_*(X)\otimes G)$

The tensor product is like tensoring the chain complex $C_*(X)$ with $0 \xrightarrow{0} G \xrightarrow{0} 0$.

Thm $\Longrightarrow$ $0 \longrightarrow H_n(X) \otimes G \longrightarrow H_n(X;G) \longrightarrow \mathrm{Tor}(H_{n-1}(X), G) \longrightarrow 0$

$$\Rightarrow H_n(X;G) \approx H_n(X) \otimes G \oplus \mathrm{Tor}(H_{n-1}(X), G)$$

Cross product on Cohomology

R: Comm. ring, $j: C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$
$(R = \mathbb{Z}, \mathbb{Z}_d, \mathbb{Q})$

For any $\varphi: C_i(X) \longrightarrow R$, $\psi: C_{n-i}(Y) \longrightarrow R$ $\Rightarrow$ $C_n(X \times Y) \xrightarrow{j} \bigoplus_i C_i(X) \otimes C_{n-i}(Y) \xrightarrow{P_i} C_i(X) \otimes C_{n-i}(Y) \xrightarrow{\boxed{\varphi \otimes \psi}} R \otimes R \longrightarrow R$

$\varphi \otimes \psi(a \otimes b) = \varphi(a) \otimes \psi(b)$

$$\Rightarrow C^i(X;R) \times C^{n-i}(Y;R) \longrightarrow C^n(X \times Y; R)$$

Furthermore:

$$\delta(\varphi \times \psi)(a \otimes b) = (\varphi \times \psi)\left(\partial a \otimes b + (-1)^i a \otimes \partial b\right) = \varphi(\partial a)\psi(b) + (-1)^i \varphi(a) \otimes \psi(\partial b)$$
$$= \left(\delta\varphi \times \psi + (-1)^i \varphi \times \delta\psi\right)(a \otimes b)$$

$\Rightarrow$ Cross product map: $H^i(X;R) \times H^{n-i}(Y;R) \longrightarrow H^n(X \times Y; R)$

$\}$ bilinear

$$H^i(X;R) \otimes_R H^{n-i}(Y,R) \longrightarrow H^n(X \times Y; R)$$

Thm If $H^i(Y,R)$ is a finitely gen. R-module for all i, then:
the cross product map gives an isom. $\bigoplus_i H^i(X;R) \otimes_R H^{n-i}(Y;R) \longrightarrow H^n(X \times Y; R)$.