Lecture 3

Wednesday, January 25, 2017 11:44 AM

CW Complexes

Def • $A \subseteq X$ is called a subcomplex if A is closed and it's a union of cells in X.

• CW pair (X, A) : CW complex X together with a subcomplex A.

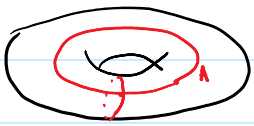
• X, Y CW complex $\Rightarrow$ $X \times Y$ CW complex

• CW pair $(X, A) \Rightarrow X/A$ inherits a CW complex str. from X.

Cells in X/A : Cells in $X - A$ together with one new 0-cell for A.

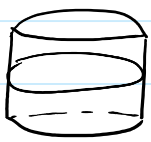
e^n_α : n cell in $X - A$ i.e. $\Phi^n_\alpha : D^n_\alpha \longrightarrow X^n$ s.t. $\Phi^n_\alpha|_{\text{int}(D^n_\alpha)} \subset X^n - A$

$\varphi^n_\alpha : S^{n-1} \longrightarrow X^{n-1} \longrightarrow X^{n-1}/A \cap X^{n-1}$

Ex  $T/A \cong S^2$ $\qquad X^n/X^{n-1} \cong \bigvee_\alpha S^n_\alpha$

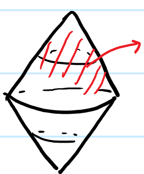
$(X^{n-1}$ connected$)$

Def Suspension X top space $\Rightarrow$ suspension of X, SX


$X \times I$

$$SX = \frac{X \times I}{(x \times \{0\} \sim x' \times \{0\},\ x \times \{1\} \sim x' \times \{1\} \mid \text{for all } x, x' \in X)}$$

$\downarrow$

SX:  $\cong CX := \dfrac{X \times I}{X \times \{0\}}$ Cone of X

• X is a CW complex $\Rightarrow$ $X \times I$ is a CW complex & $X \times \{0\}$ and $X \times \{1\}$ are subcomplexes

$I =$ •——• (1-cell; 0-cells)

$\Rightarrow CX$ and SX are CW complexes.

• Reduced suspension : Consider X together with a fixed pt $x_0 \in X$.

$x_0 \times I$: embedded line seg. in SX $\rightsquigarrow \Sigma X := \dfrac{SX}{x_0 \times \{I\}}$

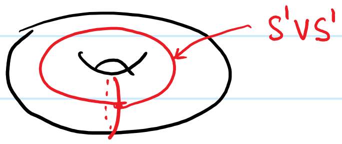
• If X is a CW complex and $x_0 \in X$ is a 0-cell $\Rightarrow \Sigma X$ inherits a CW complex str.

Ex Wedge sum, If X and Y are CW Complexes and x_0 and y_0 are 0-cells in X and Y then $X \vee Y$ inherits a CW Complex str. from X and Y.

Def Smash product: X, Y top spaces, together with fixed pts $x_0 \in X$ and $y_0 \in Y$.

$$X \vee Y \cong \{x_0\} \times Y \cup X \times \{y_0\} \subset X \times Y \Rightarrow X \wedge Y = \frac{X \times Y}{X \vee Y}$$

Similarly, if X and Y are CW Complexes and x_0 and y_0 are 0-cells in X and Y then $X \wedge Y$ inherits a Cell Complex str. from X and Y.

Ex 

$S^1 \wedge S^1 \cong S^2$ in general $S^m \wedge S^n \cong S^{n+m}$

Properties of CW Complexes:

① CW Complexes are Hausdorff.

② $A \subseteq X$ Compact Subspace, X: CW Complex $\Rightarrow$ A lies in a finite subcomplex of X.

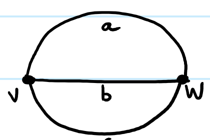
CoR: Closure of each cell intersects only finitely many other cells.

(Closure-finiteness)
Weak topology $\Rightarrow$ CW Complex

③ Any Subcomplex $A \subseteq X$ satisfies HEP.

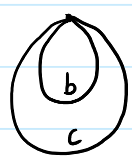
CoR For any Contractible SubComplex $A \subset X$. $X \simeq X/A$

Ex $X =$ "Theta graph"



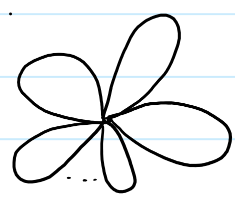
$A = a \cup \{v, w\}$: closed sub complex

$\Rightarrow X \simeq X/A$



$S^1 \vee S^1$

In general, any Connected graph with finitely many vertices and edges i.e. Finite CW Complex X s.t. $X = X^1$. is homotopy equiv. to $S^1 \vee S^1 \ldots \vee S^1$



(Collapse any edge with distinct vertices)

EX X CW complex and $x_0 \in X$ 0-cell $\Rightarrow$ $SX \simeq \Sigma X$.

④ To determine homotopy type of CW complex we only need attaching maps up to homotopy.
$S^{n-1} \subset D^n$ has HEP, thus changing attaching maps upto homotopy doesn't change the homotopy type of X^n.

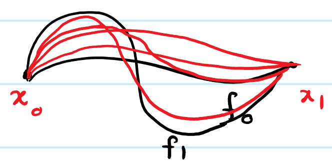
Fundamental group:

based top space $(X, x_0) \Rightarrow \pi_1(X, x_0)$: Fundamental group of X
($\pi_n(X, x_0)$: homotopy groups)

Def A path in X, $f: I \longrightarrow X$, f is a loop if $f(0) = f(1)$.

Def A homotopy of paths in X is a homotopy $f_t : I \longrightarrow X$ such that
$f_t(0) = x_0$
$f_t(1) = x_1$
are indepent of t.

If f_t is a homotopy of paths, then f_0 and f_1 are called homotopic : $f_0 \simeq f_1$

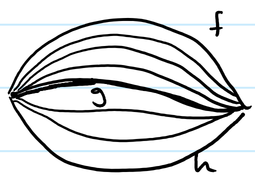


Prop $\simeq$ is an equivalence relation.
(Consequence of homotopy being an equivalence relation)

- $f \simeq f$ via $f_t = f$
- $f \simeq g$ via $f_t \rightsquigarrow g_t = f_{1-t} \rightsquigarrow g_0 = g$, $g_1 = f$
- $f \simeq g$, $g \simeq h \Rightarrow f \simeq h$

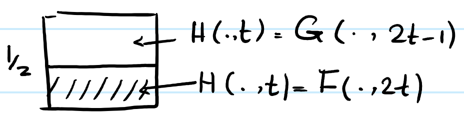
f_t : $f_0 = f$, $f_1 = g$
g_t : $g_0 = g$, $g_1 = h$

$$\Rightarrow h_t = \begin{cases} f_{2t} & 0 \leq t \leq 1/2 \\ g_{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases} \qquad \begin{matrix} h_0 = f, & h_{1/2} = g \\ h_{1/2} = g & h_1 = h \end{matrix}$$



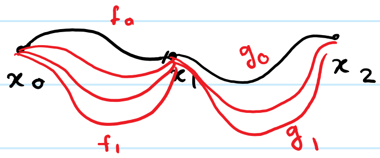
$H : I \times I \longrightarrow X$
$H(\cdot, t) = h_t(\cdot)$

is contin. follows from the fact that f_t and g_t are homotopies.



Composition of paths

$f, g : I \longrightarrow X$ s.t. $f(1) = g(0) \Rightarrow$ Composition of f and g, $f \cdot g : I \longrightarrow X$



$$f \cdot g(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

- $\underset{f_t}{f_0 \simeq f_1}$, $\underset{g_t}{g_0 \simeq g_1}$ $\Rightarrow f_0 \cdot g_0 \simeq f_1 \cdot g_1$
  $\Rightarrow f_t \cdot g_t$: homotopy btn $f_0 \cdot g_0$ and $f_1 \cdot g_1$

Def Let (X, x_0) be a based topological space. $\pi_1(X, x_0)$: the set of all homotopy classes of loops at $x_0 \in X$ i.e. $f : I \longrightarrow X$ s.t. $f(0) = f(1) = x_0$

Prop $(\pi_1(X, x_0), \cdot)$ is a group, called fundamental group of X at the base pt x_0.