Lecture 4

Sunday, January 29, 2017   7:22 PM

Properties of Composition

① $f_0 \simeq f_1$ via a homotopy f_t and $g_0 \simeq g_1$ via homotopy g_t, and $f_0(1) = f_1(1) = g_0(0) \; (= g_1(0))$

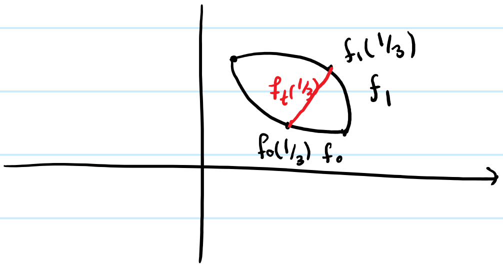
then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$

Proof $h_t = f_t \cdot g_t$ for $0 \leq t \leq 1$ $\Rightarrow$ $[f] \cdot [g] = [f \cdot g]$

② $f \cdot (g \cdot h) \simeq (f \cdot g) \cdot h$

Linear homotopies: Let f_0 and f_1 be paths in $\mathbb{R}^n$ such that $f_0(0) = f_1(0) = x_0$ and $f_1(0) = f_1(1) = x_1$. Then $f_0 \simeq f_1$.

Take $f_t = t f_1 + (1-t) f_0$, For any s, $f_t(s)$ moves along the line segment connecting $f_0(s)$ to $f_1(s)$.



Def

$f: I \to X$ is a path, a reparam. of f is $f\varphi$ where $\varphi: I \to I$ s.t. $\varphi(0) = 0$, $\varphi(1) = 1$

Lem For any path f and any reparam. $f\varphi$ we have $f \simeq f\varphi$.

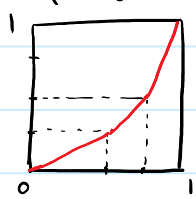
proof: Let $\varphi_t : I \to I$ for $0 \leq t \leq 1$, be $\varphi_t(s) = ts + (1-t)\varphi(s)$: homotopy btn φ and $\mathbb{1}$.

$\Rightarrow f_t := f\varphi_t$: homotopy between f and $f\varphi$.

Proof of property 2: Find a reparam. φ s.t. $(f \cdot (g \cdot h))\varphi = (f \cdot g) \cdot h$

$f \cdot (g \cdot h)$: f on $[0, 1/2]$, g on $[1/2, 3/4]$, h on $[3/4, 1]$

$(f \cdot g) \cdot h$: f on $[0, \frac{1}{4}]$, g on $[1/4, 1/2]$, h on $[\frac{1}{2}, 1]$



← graph of φ

Def $\pi_1(X, x_0)$: The set of all homotopy classes of loops at the base pt x_0

i.e. $[f]$ s.t. $f: I \to X$ w. $f(0) = f(1) = x_0$

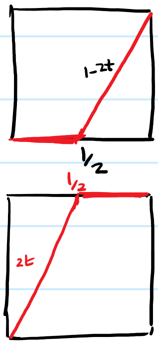
Prop $(\pi_1(X,x_0), \cdot)$ is a group.
$\uparrow$ Composition

proof ① $\Rightarrow$ $\cdot$ is welldefined.

② $\Rightarrow$ $\cdot$ is associative.

$*$ identity element: Constant path $c(t) = x_0$.

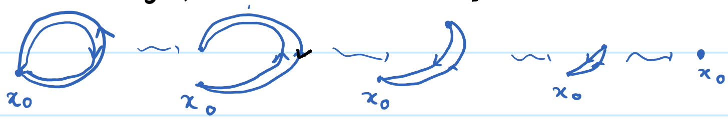
$c \cdot f \simeq f \simeq f \cdot c$ $\rightsquigarrow$ use reparametrization φ_1:



$\Rightarrow f\varphi_1 = c \cdot f \Rightarrow f \simeq c \cdot f$

φ_2: $f\varphi_2 = f \cdot c \Rightarrow f \simeq f \cdot c$

$*$ Inverse: f, $\bar{f}(s) = f(1-s)$, $f \cdot \bar{f} \simeq c \simeq \bar{f} \cdot f$



$$f_t(s) = \begin{cases} f(s) & 0 \leq s \leq 1-t \\ f(1-t) & 1-t \leq s \leq 1 \end{cases} \rightsquigarrow f_t \cdot \bar{f}_t : \text{homotopy between } f \cdot \bar{f} \text{ and } c$$

✓

Ex $\pi_1(\mathbb{R}^n, 0) = 0$

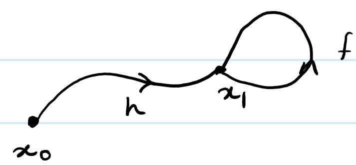
Q How does $\pi_1(X,x_0)$ depends on the choice of x_0?

$*$ $\pi_1(X,x_0)$ contains information about the path Connected Comp. of X Containing x_0.

Q what is the relation between $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ where x_0 and x_1 lie in the Same path Connected Component of X.

A: $\pi_1(X,x_0) \approx \pi_1(X,x_1)$

Let $h: I \longrightarrow X$ be a path from x_0 to x_1.



$\beta_h : \pi_1(X,x_1) \longrightarrow \pi_1(X,x_0)$

$[f] \longrightarrow [h \cdot f \cdot \bar{h}]$ well-defined, because f_t, then $h \cdot f_t \cdot \bar{h}$ is a homotopy.

Step 1 β_h is a homomorphism, $\beta_h([f \cdot g]) = [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \bar{h} \cdot h \cdot g \cdot \bar{h}]$

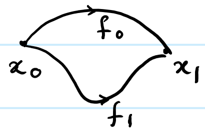
$\beta_h[f] \cdot \beta_h[g]$

Step 2 Inverse: $\beta_{\bar{h}} : \pi_1(X,x_0) \longrightarrow \pi_1(X,x_1)$ $\beta_{\bar{h}}\beta_h[f] = \beta_{\bar{h}}[h \cdot f \cdot \bar{h}] = [\bar{h} \cdot h \cdot f \cdot \bar{h} \cdot h] = [f]$

Def X is called <u>Simply Connected</u>, if X is path connected and $\pi_1(X) = 0$

Prop: X is simply connected iff there is a unique homotopy class of paths connecting any two pts.

proof ($\Rightarrow$) Simply Connected $\Rightarrow$ X is path connected $\Rightarrow$ there exist a path $f_0, f_1 : I \to X$ $\Rightarrow [f_0 \cdot \bar{f}_1] \in \pi_1(X, x_0) \Rightarrow f_0 \cdot \bar{f}_1 \simeq c$



$\Rightarrow f_0 \cdot \bar{f}_1 \cdot f_1 \simeq c \cdot f_1 \simeq f_1$

$\Rightarrow f_0 \simeq f_1$

($\Leftarrow$) all loops from x_0 (paths from x_0 to x_0) are homotopic to constant path.

$\Rightarrow \pi_1(X, x_0) = 0$

Prop $(X, x_0), (Y, y_0)$ base top space $\Rightarrow \pi_1(X \times Y, (x_0, y_0)) \approx \pi_1(X, x_0) \times \pi_1(Y, y_0)$

$[h] \in \pi_1(X \times Y, (x_0, y_0))$, $h : I \to X \times Y \rightsquigarrow h = (f, g) \Rightarrow ([f], [g]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$

homotopy $h_t \Rightarrow h_t = (f_t, g_t)$ s.t. f_t and g_t of homotopies of loops

It is a bijection between $\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

It is not hard to see that it's a homomorphism. $\Rightarrow$ isomorphism.

Prop $\begin{cases} \pi_1(S^1) = \mathbb{Z} \\ \pi_1(S^n) = 0 \quad n \geq 2 \end{cases}$

Cor 1 $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$

Cor 2 $\mathbb{R}^2$ is not homeo to $\mathbb{R}^n$ for $n \neq 2$.

$f : \mathbb{R}^2 \to \mathbb{R}^n$ homeo $\Rightarrow$ $\mathbb{R}^n \setminus \{f(0)\} \cong S^{n-1} \times \mathbb{R}$

$\mathbb{R}^2 \setminus \{0\} \cong S^1$ #z using π_1

Lemma Suppose $X = \bigcup_\alpha A_\alpha$ such that

1. for each α, A_α is path connected and open
2. " " " $x_0 \in A_\alpha$
3. " " α and β, $A_\alpha \cap A_\beta$ is path connected

Then every loop in X at x_0 is homotopic to a product of loops each of which is contained is a single A_α.

Lemma $\Rightarrow \pi_1(S^n) = 0$ for $n \geq 2$.

$A_N = S^n \setminus \{N\}$

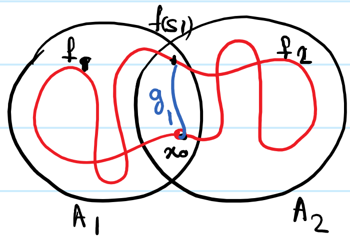
$A_S = S^n \setminus \{S\}$

$\Rightarrow S^n = \underset{\mathbb{R}^n}{\underset{\cong}{A_N}} \cup \underset{\mathbb{R}^n}{\underset{\cong}{A_S}}$, $A_N \cap A_S \cong S^{n-1} \times \mathbb{R}$: path connected

$\Rightarrow$ Every loop based at x_0 is homotopic to products of loops in A_N and A_S $\Rightarrow$ nullhomotopic

proof of lemma

Let $f: I \to X$ be a loop based at x_0.



There exists $s_0 = 0 < s_1 < s_2 < \cdots < s_m = 1$
Such that $f([s_{i-1}, s_i]) \subset A_\alpha$ for some α.

For any $s \in I$, there exist an open interval $U_s \subset I$ such that $f(\overline{U_s}) \subset A_\alpha$ for some α. $I = \bigcup_{s \in I} U_s \Rightarrow I$ is covered by finitely many of them.
$\Rightarrow$ endpoints of the intervals gave the partition.

f_i : path from $f(s_{i-1})$ to $f(s_i)$ obtained by restricting f to $[s_{i-1}, s_i]$
denote A_α containing f_i by A_i.

$f = f_1 \cdot \cdots \cdot f_m$

Let g_i be a path in $A_i \cap A_{i+1}$ connecting x_0 to $f(s_i)$.

$$\Rightarrow f \simeq \underset{\underset{A_1}{\cap}}{(f_1 \cdot \overline{g_1})} \cdot \underset{\underset{A_2}{\cap}}{(g_1 \cdot f_2 \cdot \overline{g_2})} \cdot \cdots \cdot \underset{\underset{A_m}{\cap}}{(g_{m-1} \cdot f_m)}$$

Induced homomorphism

Let (X, x_0), (Y, y_0) be based top. spaces and $\varphi: X \to Y$ s.t. $\varphi(x_0) = y_0$.

$\Rightarrow \varphi_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $f: I \to X \xrightarrow{\varphi} Y$

$\varphi_*([f]) = [\varphi f]$

well-defined : homotopy f_t btn f_0 and f_1 $\Rightarrow$ φf_t : homotopy btn φf_0 and φf_1

homomorphism : $\varphi_*([f \cdot g]) = [\varphi(f \cdot g)] = [\varphi f \cdot \varphi g] = [\varphi f] \cdot [\varphi g] = \varphi_* f \cdot \varphi_* g$

$$\varphi(f \cdot g(t)) = \begin{cases} \varphi(f(2t)) & 0 \leq t \leq 1/2 \\ \varphi g(1-2t)) & 1/2 \leq t \leq 1 \end{cases}$$

Properties :

- $(X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\psi} (Z, z_0)$ then $\psi_* \varphi_* = (\psi\varphi)_*$
- $\mathbb{1}_* = \mathbb{1} \leftarrow \mathbb{1}: \pi_1(X, x_0) \to \pi_1(X, x_0)$
  $\hookrightarrow \mathbb{1}: X \to X$

In particular, $\overset{x_0 \in}{A} \underset{\text{subspace}}{\subset} X \Rightarrow i: A \hookrightarrow X$ induces $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$

Prop ① X retracts onto $A \Rightarrow i_*$ is injective.

pf $r: X \longrightarrow A$ s.t. $ri = \mathbb{1} \Rightarrow r_* i_* = \mathbb{1} \Rightarrow i_*$: injective

Cor S^1 is not a retract of D^2, because $\pi_1(S^1) \approx \mathbb{Z}$, $\pi_1(D^2) = 0$ no injective map from $\mathbb{Z}$ to 0.

② If A is a deformation retraction of X, then i_* is isomorphism.

pf we should show that i_* is surjective. Let $[f] \in \pi_1(X, x_0)$ and r_t be corresponding defor. retrac. Then $r_t f$ is a homotopy btn f and $r_1 f \subset A \rightsquigarrow i_*[r_1 f] = [f] \Rightarrow$ surjective

Prop If $\varphi: X \longrightarrow Y$ is a homotopy equivalence, then

$$\varphi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \varphi(x_0))$$

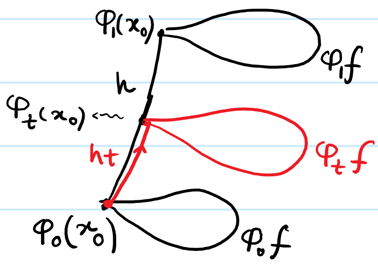
is an isomorphism for every $x_0 \in X$.

Lemma Let $\varphi_t: X \longrightarrow Y$ is a homotopy and h is the path $\varphi_t(x_0)$.

$\Rightarrow \varphi_{0*} = \beta_h \varphi_{1*}$

$$\begin{tikzcd}
 & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\beta_h"] \\
\pi_1(X, x_0) \arrow[ur, "\varphi_{1*}"] \arrow[dr, "\varphi_{0*}"'] & \\
 & \pi_1(Y, \varphi_0(x_0))
\end{tikzcd}$$

pf



$h_t(s) = h(ts)$

$\Rightarrow h_t(\varphi_t f)\bar{h}_t$: homotopy from $\varphi_0 f$ to $h \cdot \varphi_1 f \cdot \bar{h}$

Proof of prop:

$\psi: Y \longrightarrow X$, $\varphi\psi \simeq \mathbb{1} \Rightarrow \underset{\varphi_*\psi_*}{(\varphi\psi)_*}$: isomorphism $\Rightarrow \psi_*$: injective, φ_* : surjective

$\psi\varphi \simeq \mathbb{1} \Rightarrow \psi_* \varphi_*$: iso $\Rightarrow \varphi_*$: injective

$\Rightarrow \varphi_*$: isomorphism