Lecture 5

Wednesday, February 1, 2017 11:17 AM

Part 1 Invariance of fundamental group under homeo and homotopy equiv.

Part 2 $\pi_1(S^1) \approx \mathbb{Z}$

Induced homomorphisms

Let (X,x_0), (Y,y_0) be based top. spaces and $\varphi: X \longrightarrow Y$ s.t. $\varphi(x_0)=y_0$.

$\Rightarrow \varphi_*: \pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0)$, $f: I \longrightarrow X \xrightarrow{\varphi} Y$

$\varphi_*([f]) = [\varphi f]$

well-defined: $[f_0]=[f_1]$, f_t homotopy btn f_0 and f_1 $\Rightarrow$ φf_t: homo. btn φf_0 and φf_1 $\Rightarrow [\varphi f_0] = [\varphi f_1]$

homomorphism: $\varphi_*([f \cdot g]) = [\varphi(f \cdot g)] = [\varphi f \cdot \varphi g] = [\varphi f] \cdot [\varphi g] = \varphi_* f \cdot \varphi_* g$

$$\varphi(f \cdot g(t)) = \begin{cases} \varphi f(2t) & 0 \leq t \leq 1/2 \\ \varphi g(1-2t) & 1/2 \leq t \leq 1 \end{cases}$$

Properties:

① $(X,x_0) \xrightarrow{\varphi} (Y,y_0) \xrightarrow{\psi} (Z,z_0)$ then $\psi_* \varphi_* = (\psi\varphi)_*$

because $\psi_* \varphi_*([f]) = \psi_*([\varphi f]) = [\psi(\varphi f)] = [(\psi\varphi) f] = (\psi\varphi)_*$

② $\mathbb{1}: (X,x_0) \longrightarrow (X,x_0) \Rightarrow \mathbb{1}_* = \mathbb{1}$, $\mathbb{1}: \pi_1(X,x_0) \longrightarrow \pi_1(X,x_0)$

Cor If $\varphi: (X,x_0) \longrightarrow (Y, y_0 = \varphi(x_0))$ is a homeo. then φ_* is an isomorphism

Pf Let $\psi: (Y,y_0) \longrightarrow (X,x_0)$ s.t. $\varphi\psi = \mathbb{1}$ and $\psi\varphi = \mathbb{1}$

$\xRightarrow[①,②]{} \varphi_* \psi_* = \mathbb{1}$, $\psi_* \varphi_* = \mathbb{1}$

Special Case: Let $\underset{\text{subspace}}{A \subset X}$ be a deformation retraction of X.

- For any subspace $x_0 \in A \subset X$, the inclusion map $i: (A,x_0) \hookrightarrow (X,x_0)$ induces $i_*: \pi_1(A,x_0) \longrightarrow \pi_1(X,x_0)$.

Prop ① If X retracts onto A $\Rightarrow i_*$ is injective.

Pf $r: X \longrightarrow A$ s.t. $ri = \mathbb{1} \Rightarrow r_* i_* = \mathbb{1} \Rightarrow i_*$: injective

__Cor__ S^1 is not a retract of D^2, because $\pi_1(S^1) \approx \mathbb{Z}$, $\pi_1(D^2)=0$ no injective map from $\mathbb{Z}$ to 0.

② If A is a deformation retraction of X, then i_* is isomorphism.

__pf__ we should show that i_* is surjective. Let $[f] \in \pi_1(X, x_0)$ and r_t be Corresponding defor. retrac. Then $r_t f$ is a homotopy btn f and $r_1 f \subset A$ $\rightsquigarrow$ $i_*[r_1 f] = [f]$ $\Rightarrow$ Surjective

__Prop__ If $\varphi: X \longrightarrow Y$ is a homotopy equivalence, then

$$\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0 = \varphi(x_0))$$

is an isomorphism, for any $x_0 \in X$.

__Lemma__ Let $\varphi_t : X \longrightarrow Y$ is a homotopy and h is the path $\varphi_t(x_0)$.

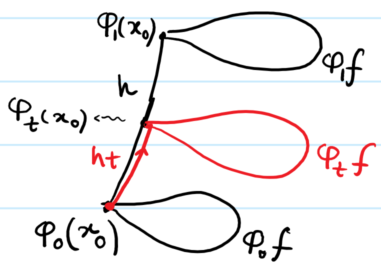
$\Rightarrow$ $\varphi_{0*} = \beta_h \varphi_{1*}$

$$\begin{tikzcd} & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\beta_h"] \\ \pi_1(X, x_0) \arrow[ur, "\varphi_{1*}"] \arrow[r, "\varphi_{0*}"'] & \pi_1(Y, \varphi_0(x_0)) \end{tikzcd}$$

__pf__

$h_t(s) = h(ts)$

$\Rightarrow$ $h_t(\varphi_t f)\bar{h}_t$: homotopy from $\varphi_0 f$ to $h \cdot \varphi_1 f \cdot \bar{h}$



__Proof of prop__:

$$(X, x_0) \xrightarrow{\varphi} (Y, y_0 = \varphi(x_0)) \xrightarrow{\psi} (X, \psi(y_0)) \xrightarrow{\varphi} (Y, \varphi(\psi(y_0)))$$

$\psi\varphi \simeq \mathbb{1}$ $\Rightarrow$ $\psi_*\varphi_* = \underbrace{\beta_h}_{\text{isomorphism}}$ $\rightsquigarrow$ $\psi_*\varphi_*$: iso $\Rightarrow$ φ_* : injective, ψ_* : surjective

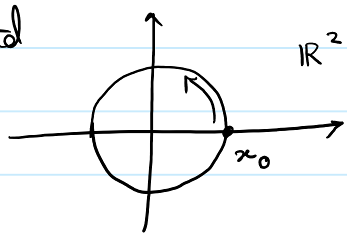
$\varphi\psi \simeq \mathbb{1}$ $\Rightarrow$ $\varphi_*\psi_*$: isomorphism $\rightsquigarrow$ ψ_* : injective

$\rightsquigarrow$ ψ_* : isomorphism $\rightsquigarrow$ φ_* : isomorphism

__Part 2__

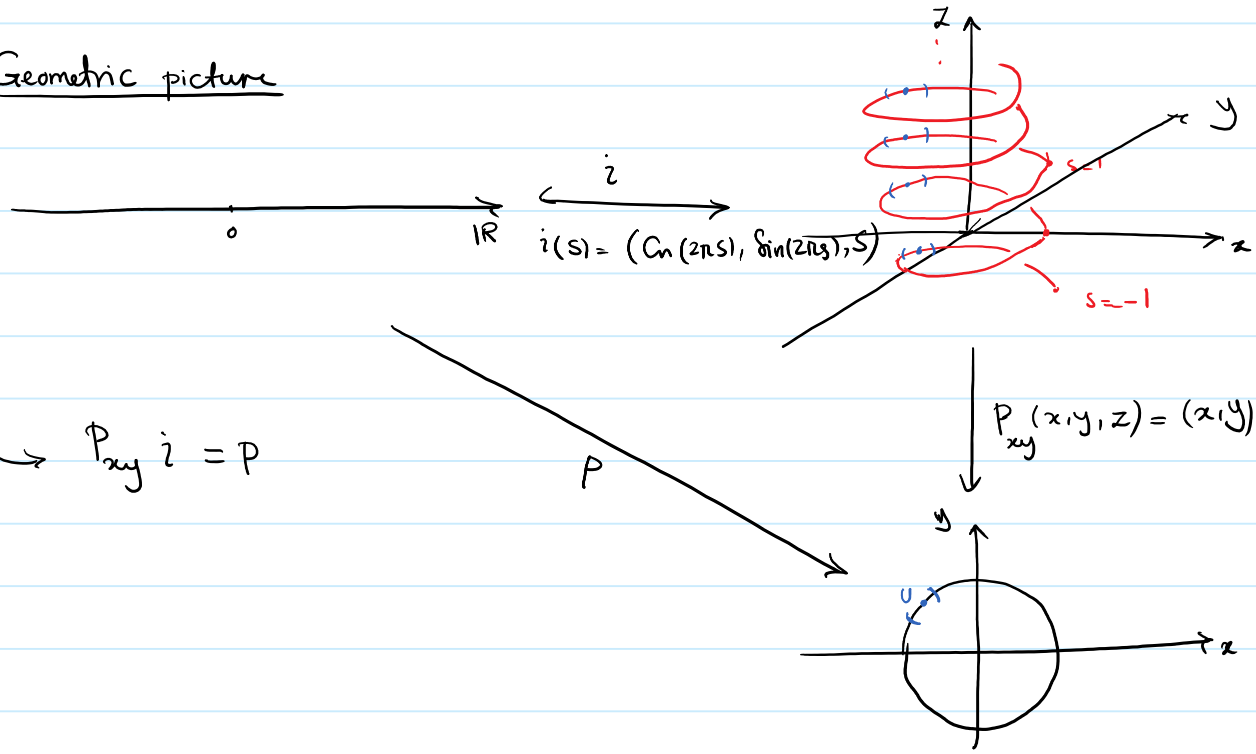
__Thm__ $\pi_1(S^1, x_0)$ is an infinite cyclic group, generated by $\omega : I \longrightarrow S^1$

$$\omega(s) = (\cos(2\pi s), \sin(2\pi s))$$



Let $p: \mathbb{R} \longrightarrow S^1$ be the map $p(s) = (\cos(2\pi s), \sin(2\pi s))$.

<u>Geometric picture</u>



$\rightsquigarrow$ $P_{xy}\, i = p$

<u>Def</u> A <u>covering space</u> of X, is a space $\tilde{X}$, together with a map $p: \tilde{X} \longrightarrow X$ such that for each $x \in X$, there exist a nbd U of x in X for which $p^{-1}(U)$ is a disjoint union of open sets each of which is mapped homeo. onto U by p.

Let $p: \tilde{X} \longrightarrow X$ be a covering space.

<u>Def</u> A lifting of a map $f: Y \longrightarrow X$ is a map $\tilde{f}: Y \longrightarrow \tilde{X}$ such that $p\tilde{f} = f$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

<u>Homotopy Lifting property</u>: Given a homotopy $f_t: Y \longrightarrow X$ and a lifting $\tilde{f}_0: Y \longrightarrow \tilde{X}$ of f_0, then there exist a unique homotopy $\tilde{f}_t: Y \longrightarrow \tilde{X}$ of $\tilde{f}_0$ that lifts f_t i.e. $p\tilde{f}_t = f_t$.

<u>Cor</u> For any pt $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$ and any path $f: I \longrightarrow X$ starting at x_0, there exists a unique lift $\tilde{f}: I \longrightarrow \tilde{X}$ of f starting at $\tilde{x}_0$.

<u>Pf</u> Let $Y = \{p\}$, $f_0: Y \longrightarrow X$, $f_t: Y \longrightarrow X$ $\Rightarrow$ There exsit a unique lift: $\tilde{f}_t: Y \longrightarrow \tilde{X}$

$f_0(p) = x_0$ $\qquad$ $f_t(p) = f(t)$

$\Rightarrow$ define $\tilde{f}(t) = \tilde{f}_t(p)$

Cor 2 For any $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$, any homotopy $f_t : I \longrightarrow X$ of paths starting at x_0, there exists a unique lifted homotopy $\tilde{f}_t : I \longrightarrow \tilde{X}$ of paths starting at $\tilde{x}_0$.

Pf $f_0 : I \longrightarrow X \xrightarrow[\text{Cor 1}]{} $ unique lift $\tilde{f}_0 : I \longrightarrow \tilde{X}$ starting at $\tilde{x}_0$.

$\xrightarrow[\text{HLP}]{}$ unique lift $\tilde{f}_t : I \longrightarrow \tilde{X}$ of f_t starting at $\tilde{x}_0$.

Pf of thm For any integer n, let $\omega_n : I \longrightarrow S^1$ be the loop
$$\omega_n(s) = (\cos(2\pi n s), \sin(2\pi n s))$$

Ex Show that $[\omega_n] = [\omega]^n$ i.e. $\omega_n \simeq \underbrace{\omega . \omega . \dots . \omega}_{n\text{-times}}$

The unique lift of ω_n to $\mathbb{R}$ starting at 0 is $\tilde{\omega}_n(s) = s$.

- If $n \neq m$ then $\omega_n \not\simeq \omega_m$. Suppose f_t is a homotopy btn $f_0 = \omega_n$ and $f_1 = \omega_n$.

Then Cor 2 implies, there exist a unique lifting $\tilde{f}_t$ of f_t starting at 0.

$\Rightarrow \tilde{f}_0 = \tilde{\omega}_n$, $\tilde{f}_1 = \tilde{\omega}_m \Rightarrow \tilde{\omega}_n(1) = \tilde{\omega}_m(1) \Rightarrow n = m$ Contradiction! $\Rightarrow \omega_n \not\simeq \omega_m$

Consider an element $[f] \in \pi_1(S^1, x_0)$, $f : I \longrightarrow S^1$ is a loop base at x_0

$\xrightarrow[\text{Cor 1}]{}$ There exist a lift $\tilde{f}$ of f starting at $0 \in p^{-1}(x_0)$.

$p\tilde{f} = f \rightsquigarrow p\tilde{f}(1) = f(1) = x_0 \rightsquigarrow \tilde{f}(1) \in p^{-1}(x_0) \rightsquigarrow \tilde{f}(1)$ is an integer.

let : $n = \tilde{f}(1)$, another path $\tilde{\omega}_n : I \longrightarrow \mathbb{R}$
$$\tilde{\omega}_n(s) = ns$$

$\tilde{f}$ is homotopic to $\tilde{\omega}_n$ via linear homotopies, $(1-t)\tilde{f} + t\tilde{\omega}_n = \tilde{f}_t$

$\Rightarrow p\tilde{f}_t$: homotopy btn f and $\overset{\omega_n =}{} p\tilde{\omega}_n(s) = (\cos(2\pi n s), \sin(2\pi n s))$

$\boxtimes$