Lecture 6

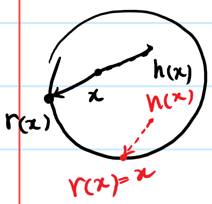
Sunday, February 5, 2017 5:40 PM

Recall
- Fundamental group $\pi_1(X, x_0)$
- Invariant under homotopy equiv.
- $\pi_1(S^1) \approx \mathbb{Z}$, $\pi_1(S^n) = 0$ for $n \geqslant 2$

Part I Application of $\pi_1(S^1) \approx \mathbb{Z}$

Thm: (Brouwer fixed pt thm) Every conti map $h: D^2 \longrightarrow D^2$ has a fixed pt.
i.e. there exists $x \in D^2$ s.t. $h(x) = x$.

Proof by Contradiction: Assume $h: D^2 \to D^2$ has no fixed pt. Define
$r: D^2 \longrightarrow S^1$ as in the picture:



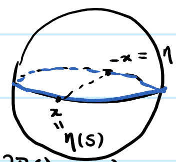
If $x \in S^1$, $r(x) = x$ $\Rightarrow$ r is a retraction.
From last session $\rightsquigarrow$ $\underset{\text{induced by inclusion}}{i_*}: \underset{\mathbb{Z}}{\pi_1(S^1)} \longrightarrow \underset{0}{\pi_1(D^2)}$ is injective. $\times$

Thm: (Borsuk_Ulam thm) For any conti. map $f: S^2 \longrightarrow \mathbb{R}^2$, there exists antipodal pts x and $-x$ with $f(x) = f(-x)$.

proof by Contradition: Suppose $f: S^2 \to \mathbb{R}^2$ is conti. and for any $x \in S^2$, $f(x) \neq f(-x)$.
Thus $g: S^2 \longrightarrow S^1$

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

$[0,1] \xrightarrow{\eta}$



$\xrightarrow{g} S^1$ $\quad h = g\eta: I \longrightarrow S^1$ loop in S^1

$\eta(s) = (\cos(2\pi s), \sin(2\pi s), 0)$

- The loop η is homotopically trivial in S^2 $\Rightarrow$ $g\eta$: homotopically trivial.
- $g(x) = -g(-x)$ $\rightsquigarrow$ $h(s + 1/2) = -h(s)$ Let $\tilde{h}: [0,1] \longrightarrow \mathbb{R}$ be a lifting of h.
  i.e. $p\tilde{h} = h$ $\rightsquigarrow$ $p\tilde{h}(s + \frac{1}{2}) = h(s + \frac{1}{2}) = -h(s) = -p\tilde{h}(s)$

$$\Rightarrow \underbrace{\tilde{h}(s + \tfrac{1}{2}) - \tilde{h}(s)}_{\text{Conti. function of } s} = \frac{q}{2} \leftarrow \text{odd number}$$

$\Rightarrow q$ is constant for any $0 \leqslant s \leqslant 1/2$

$$\Rightarrow \tilde{h}(1) = \tilde{h}(\tfrac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + \underset{\text{odd}}{q} \neq 0 \rightsquigarrow [h] = [\omega_q] \ \times$$

Rmk We may use homology to prove Brouwer fixedpt. thm and Borsuk-Ulam thm in higher dimension.

Cor If $S^2 = \bigcup_{i=1}^{3} A_i$ where A_1, A_2, A_3 are closed, then at least one A_i contain a pair of antipodal pts i.e. $\{x, -x\} \subseteq A_i$ for a $x \in S^2$.

Pf Define $f=(d_1,d_2): S^2 \longrightarrow \mathbb{R}^2$

$d_i(x)=\inf_{y\in A_i}|x-y|$ distance of x from A_i

$\overset{\text{BUT}}{\Longrightarrow}$ There exists $x\in S^2$ s.t. $f(x)=f(-x)$

$\Rightarrow d_1(x)=d_1(-x)$

$d_2(x)=d_2(-x)$

If $d_1(x)=d_1(-x)=0$ $(\overset{\text{OR}}{d_2(x)}=d_2(-x)=0) \Rightarrow x,-x\in A_1$ (or $x,-x\in A_2$)

If $d_1(x)=d_1(-x)\neq 0$, $d_2(x)=d_2(x)\neq 0$ $\Rightarrow x,-x\in A_3$

Thm (Fundamental thm of algebra): Every polynomial of positive degree and with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

Pf by Contradition Assume $P(Z)=Z^n+a_1Z^{n-1}+\cdots+a_n$ with $n>0$ has no root.

$\Rightarrow P(Z)\neq 0$ for $Z\in\mathbb{C}$.

family of loops in S^1 based at 1, $r\geq 0$: $f_r(s)=\dfrac{P(re^{2\pi is})/P(r)}{|P(re^{2\pi is})|/|P(r)|}$ $f_0(s)=1$ Constant loop

$\Rightarrow$ For any $r\geq 0$, f_r is homotopically trivial.

For $r=|z|>\max(|a_1|+\cdots+|a_n|, 1)$

$$|z|^n>(|a_1|+\cdots+|a_n|)|z|^{n-1}>|a_1||z|^{n-1}+|a_2|z^{n-2}+\cdots+|a_n|\geq|a_1z^{n-1}+\cdots+a_n|$$

$\rightsquigarrow P_t(z)=z^n+t(a_1z^{n-1}+\cdots+a_n)$: has no root for $|z|=r$ and $0\leq t\leq 1$

For such r, we use P_t to construct a homotopy btn f_r and ω_n:

$f_r^t(s)=\dfrac{P_t(re^{2\pi is})/P_t(r)}{|P_t(re^{2\pi is})|/|P_t(r)|}$: homotopy btn $f_r^1(s)=f_r(s)$ and

$$f_r^0(s)=\frac{\cancel{r^n}e^{2\pi ins}/\cancel{r^n}}{|r^ne^{2\pi ins}|/r^n}=e^{2\pi ins}=\omega_n(s)$$

$\Rightarrow [f_r]=[\omega_n]$ $\times$.

Part 2: Van Kampen's thm:

Free product of group

Let $\{G_\alpha\}_{\alpha\in I}$ be a collection of group.

Def. A word in $\{G_\alpha\}_{\alpha\in I}$ is a sequence $g_1g_2\cdots g_m$ of finite length s.t. $g_i\in G_{\alpha_i}$ for some $\alpha_i\in I$.

- A word $g_1g_2\cdots g_m$ is called reduced if for any i $\alpha_i\neq \overset{\text{identity}}{e_{\alpha_i}}\in G_{\alpha_i}$ and $\alpha_i\neq\alpha_{i+1}$.

Any word can be simplified to a "reduced words" by:

① Remove g_i if $g_i = e_{\alpha_i} \in G_{\alpha_i}$

② If $\alpha_i = \alpha_{i+1}$ then replace the two letter $g_i g_{i+1}$ by the multi $g_i \cdot g_{i+1} \in G_{\alpha_i}$

<u>Def</u> The free group $*_\alpha G_\alpha$ as a set consists of the reduced words in $\{G_\alpha\}_{\alpha \in I}$.

Multiplication: $(g_1 g_2 \cdots g_m)(g_1' g_2' \cdots g_n') = g_1 g_2 \cdots g_m g_1' g_2' \cdots g_n'$

then make it reduced by the above operations ① and ②.

For example, $(g_1 g_2 \cdots g_m)^{-1} = g_m^{-1} g_{m-1}^{-1} \cdots g_1^{-1}$, $(g_1 g_2 \cdots \cancel{g_m} \cancel{g_m^{-1}} \cancel{g_{m-1}^{-1}} \cdots g_1^{-1} = e)$

identity: empty word

<u>Properties</u> ① Every G_α is naturaly identified with a <u>subgroup</u> of G. Consisting of empty word and one-litter words $g \in G_\alpha$ where $g \neq e_\alpha$.

② For any group H, any collection of homo $\varphi_\alpha : G_\alpha \to H$ extends to a homo $\varphi : *_\alpha G_\alpha \longrightarrow H$ as

$$\varphi(g_1 \cdots g_n) = \varphi_{\alpha_1}(g_1)\varphi_{\alpha_2}(g_2) \cdots \varphi_{\alpha_n}(g_n)$$

$g_i \in G_{\alpha_i}$

<u>Ex</u> $\overset{a}{\mathbb{Z}} * \overset{b}{\mathbb{Z}}$, elements of the form $a b^2 a^{-3} b^5$: reduced word

$$\frac{(ab^2)(b^{-1}a^3) = a\, ba^3}{(ab)^2}$$

<u>EX</u> $\underset{a}{\mathbb{Z}_2} * \underset{b}{\mathbb{Z}_2}$ $\quad a^2 = e,\ b^2 = e,\ \mathbb{Z}_2 * \mathbb{Z}_2 \overset{\text{set}}{=} \{a, b, ab, aba, abab, \ldots, \underset{(ab)^{-1}}{ba}, \underset{a(ab)^2}{bab}, \underset{(ab)^{-2}}{baba}, \ldots\} \cup \{\text{empty word}\}$

$ba = (ab)^{-1}$

$\mathbb{Z} \rtimes \mathbb{Z}_2 = \mathbb{Z}_2 * \mathbb{Z}_2$: infinite cyclic subgroup gen. by ab : $\mathbb{Z}$

$a(ab)a^{-1} = ba = (ab)^{-1}$ $\quad$ A subgroup iso to $\mathbb{Z}_2$ gen. by a

<u>Def</u> Free group $G \approx$ free product of any number of $\mathbb{Z}$ i.e. there exists a family $\{a_\alpha\}$ of elements of G such that each a_α generates an infinite cyclic subgroup G_α of G ($G_\alpha \approx \mathbb{Z}$) and $G = *_\alpha G_\alpha$. $\quad G = \langle a_\alpha \rangle_{\alpha \in I}$

<u>Van Kampen thm</u>

Let $X = \bigcup_\alpha A_\alpha$ such that each A_α is open and path connected. Furthermore, $x_0 \in \bigcap_\alpha A_\alpha$. Let $i_\alpha : \pi_1(A_\alpha) \longrightarrow \pi_1(X)$ be the homo induced by inclusion $(A, x_0) \hookrightarrow (X, x_0)$.

Then, these homos induce a homo $\Phi : *_\alpha \pi_1(A_\alpha) \longrightarrow \pi_1(X)$.

① If for any α and β, $A_\alpha \cap A_\beta$ is path connected, then Φ is surjective.
(Lemma 1.15)

For any α and β :

$$\begin{tikzcd} & \pi_1(A_\alpha) \arrow[dr, "i_\alpha"] & \\ \pi_1(A_\alpha \cap A_\beta) \arrow[ur, "i_{\alpha\beta}"] \arrow[dr, "i_{\beta\alpha}"'] & & \pi_1(X) \\ & \pi_1(A_\beta) \arrow[ur, "i_\beta"'] & \end{tikzcd}$$

$i_\alpha i_{\alpha\beta} = i_\beta i_{\beta\alpha}$: both are the induced maps by $A_\alpha \cap A_\beta \hookrightarrow X$.

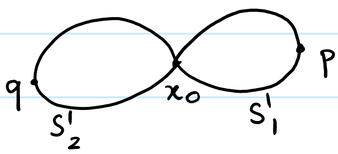
$\Rightarrow$ For any $\omega \in \pi_1(A_\alpha \cap A_\beta)$, $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1} \in *_\alpha \pi_1(A_\alpha)$

$$\Phi\left(i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}\right) = i_\alpha i_{\alpha\beta}(\omega)\, i_\beta\left(i_{\beta\alpha}(\omega)^{-1}\right) = e$$

(2) If for any α, β, γ , $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected , then $\mathrm{Ker}(\Phi)$ is the normal subgroup generated by the elements of the form $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$.

$\Rightarrow \pi_1(X) \approx *_\alpha \pi_1(A_\alpha) / N$

<u>Ex</u> $X = S^1 \vee S^1$



$A_p = X \setminus \{p\}$

$A_q = X \setminus \{q\}$

$\Rightarrow A_p$ (A_q) deform. retracts on S^1_2 (S^1_1)

$A_p \cap A_q$: deformation retracts on $\{x_0\}$.

$\Rightarrow \pi_1(X, x_0) = \pi_1(S^1_1, x_0) * \pi_1(S^1_2, x_0) \approx \mathbb{Z} * \mathbb{Z}$

In general, $\pi_1(\vee_\alpha S^1_\alpha) \approx *_\alpha \mathbb{Z}^\alpha$

<u>Cor</u> Any connected graph G is homotopy equiv. to wedge sum of a collection of circles

$\Rightarrow \pi_1(G)$ is a free group .

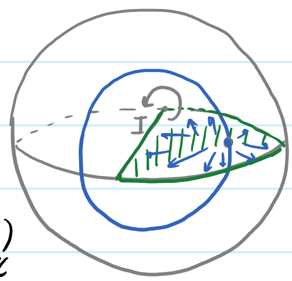
More generally, Let $\{(X_\alpha, x_\alpha)\}_{\alpha \in I}$ be a collection of based topological spaces such that every X_α contains a nbd U_α of x_α which deform. retracts on x_α . Then

$X = \vee_\alpha X_\alpha = \dfrac{\coprod_\alpha X_\alpha}{(x_\alpha \sim x_\beta \mid \alpha, \beta \in I)}$ $\Rightarrow \pi_1(X, x_0) = *_\alpha \pi_1(X_\alpha, x_\alpha)$

Take $A_\alpha = X_\alpha \cup \bigcup_{\beta \neq \alpha} U_\beta \subset X$.

$A_\alpha \cap A_\beta = \bigcup_{\alpha \in I} U_\alpha \subset X$ deform. retracts on x_0 .

<u>Ex</u> $X = \mathbb{R}^3 \setminus A$



$\mathbb{R}^3 \setminus D^3$: deform retracts on ∂D^3

$D^3 \setminus A$: deform retracts on $\partial D^3 \cup I$

$\Rightarrow X$ deform retracts $\partial D^3 \cup I \simeq S^2 \vee S^1 \Rightarrow \pi_1(X) \approx \pi_1(S^2) * \pi_1(S^1) \approx \mathbb{Z}$