Lecture 7

Wednesday, February 8, 2017 8:34 AM

__Van Kampen thm__: Suppose $X = \bigcup_\alpha A_\alpha$, such that every A_α is open, path connected and $x_0 \in A_\alpha$.
If for any α and β, $A_\alpha \cap A_\beta$ is path connected then

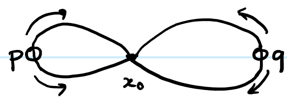
$$\Phi : *_\alpha \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective. Note that Φ is the extension of the homo.s $j_\alpha : \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$ induced by inclusion $(A_\alpha, x_0) \hookrightarrow (X, x_0)$.
If every $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then $\ker\Phi$ is the normal subgroup generated by all elements of the form $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$.

$$\begin{tikzcd} & \pi_1(A_\alpha, x_0) \arrow[dr, "i_\alpha"] & \\ \pi_1(A_\alpha \cap A_\beta, x_0) \arrow[ur, "i_{\alpha\beta}"] \arrow[dr, "i_{\beta\alpha}"'] & & \pi_1(X, x_0) \\ & \pi_1(A_\beta, x_0) \arrow[ur, "i_\beta"'] & \end{tikzcd}$$

__Special Case__ If for all α, β, $A_\alpha \cap A_\beta$ is simply connected $\Rightarrow$ $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$ is an isomorphism.

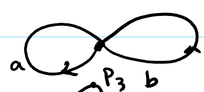
__EX__: $X = S^1 \vee S^1$



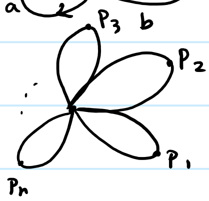
$A_p = X - \{p\} \simeq S^1$
$A_q = X - \{q\} \simeq S^1$
$A_p \cap A_q \simeq \{x_0\}$ ⇝ simply connected

$\Rightarrow \Phi : \pi_1(A_p) * \pi_1(A_q) \longrightarrow \pi_1(X)$ isomorphism

⇝ $\pi_1(X) \approx \underset{\langle b\rangle}{\mathbb{Z}} * \underset{\langle a\rangle}{\mathbb{Z}}$



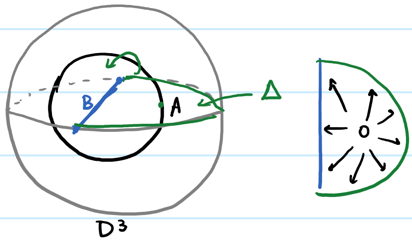
In general, $X = \underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{n \text{ time}}$



$\Rightarrow \pi_1(X) \approx \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n\text{-times}}$

__Prop__ Let $(X, x_0) = \bigvee_\alpha (X_\alpha, x_\alpha) = \dfrac{\coprod_\alpha X_\alpha}{(x_\alpha \sim x_\beta : \text{any } \alpha \text{ and } \beta)}$. If any X_α contains a nbd U_α of x_α which deform retracts on x_α then $\pi_1(X, x_0) \approx *_\alpha \pi_1(X_\alpha, x_\alpha)$

Ex $X = \mathbb{R}^3 \setminus A$

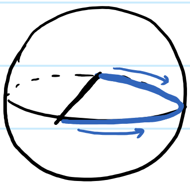


$\mathbb{R}^3 \setminus \text{int}(D^3)$ deform retracts on ∂D^3.

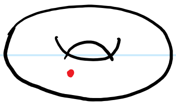
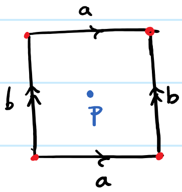
$\Rightarrow \mathbb{R}^3 \setminus A \simeq D^3 \setminus A$ or $\pi_1(D^3 \setminus A) \xrightarrow[\text{isom.}]{} \pi_1(\mathbb{R}^3 \setminus A)$

A intersects Δ in a pt, $\Delta \setminus A \cap \Delta$ deform retracts on $\partial \Delta$.

$\Rightarrow D^3 \setminus A$ deform retracts on $\partial D^3 \cup B$.

 $\rightsquigarrow$ Contract blue to one pt 

$\therefore S^1 \vee S^2 \rightsquigarrow \pi_1(\mathbb{R}^3 \setminus A) \approx \underbrace{\pi_1(S^1)}_{\overset{\mathbb{Z}}{\shortparallel}} * \cancel{\pi_1(S^2)}^{\,0}$
$\approx \mathbb{Z}$

Ex $T =$  $\rightsquigarrow$ 

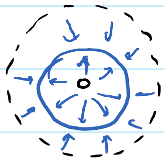
$A_1 = T \setminus \{p\} \simeq S^1 \vee S^1$

$A_2 = \text{Interior of 2-cell} = T - X^1$

$\simeq pt$

$\rightsquigarrow A_1 \cap A_2 = A_2 \setminus \{p\} \simeq S^1$

$\pi_1(A_1 \cap A_2) \longrightarrow \pi_1(A_1)$



$\Phi : \underset{\mathbb{Z} * \mathbb{Z}}{\pi_1(A_1)} * \underset{0}{\pi_1(A_2)} \longrightarrow \pi_1(T)$ $\rightsquigarrow$ $\Phi : \overset{\langle a \rangle}{\mathbb{Z}} * \overset{\langle b \rangle}{\mathbb{Z}} \longrightarrow \pi_1(T)$

$\ker \Phi$ is generated by : $aba^{-1}b^{-1}$

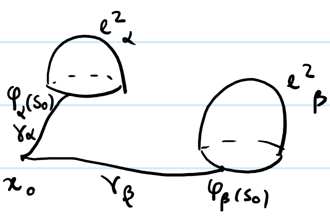
$\Rightarrow \pi_1(T) \approx \overset{\langle a \rangle}{\mathbb{Z}} * \overset{\langle b \rangle}{\mathbb{Z}} \big/ \langle [a,b] \rangle = \mathbb{Z} \times \mathbb{Z}$

In general,

Prop Let Y be a topological space obtained from attach a collection of 2-cells e^2_α to a path connected space X by attaching maps $\varphi_\alpha : S^1 \longrightarrow X$.
Choose $x_0 \in X$. Then,

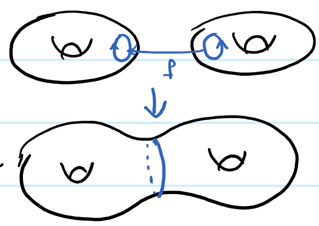
(a) The inclusion $(X, x_0) \hookrightarrow (Y, x_0)$ induces a surjective homo $\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0)$

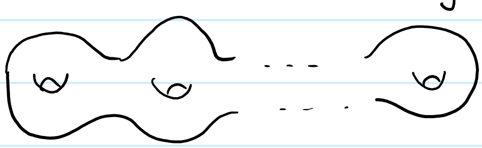
(b) Pick a base pt $s_0 \in S^1$ together with a path γ_α from x_0 to $\varphi_\alpha(s_0)$.



Then $\ker(\varphi_*)$ is the normal subgroup gene. by all $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$ for $\alpha \in I$.

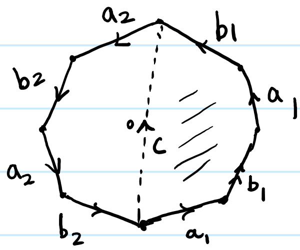
Ex Oriented surface of genus g : $M_g = \underbrace{T \# T \# \cdots \# T}_{g \text{ times}}$

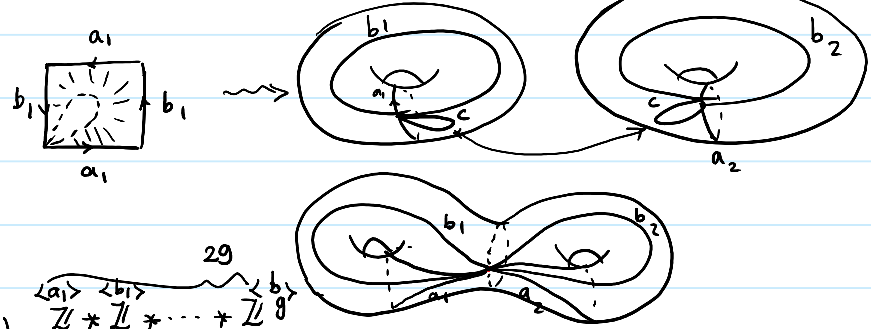




- Take a $4g$ gon, and identify the edges as follows:

$\underline{g=2}$



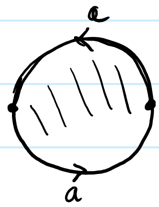


- one 0-cell
- 4 1-cells
- one 2-cell

$$\rightsquigarrow \pi_1(M_g) = \frac{\overbrace{\underset{\langle a_1\rangle}{\mathbb{Z}} * \underset{\langle b_1\rangle}{\mathbb{Z}} * \cdots * \underset{\langle b_g\rangle}{\mathbb{Z}}}^{2g}}{\langle [a_1,b_1][a_2,b_2]\cdots[a_g,b_g]\rangle}$$

(not Commutative)

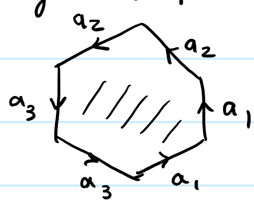
$\underline{\text{Ex}}$ $\mathbb{RP}^2 = \dfrac{D^2}{(x\sim -x \mid x\in\partial D^2)} \rightsquigarrow$



$$\rightsquigarrow \pi_1(\mathbb{RP}^2) = \frac{\langle a\rangle}{\langle a^2\rangle} \approx \mathbb{Z}_2$$

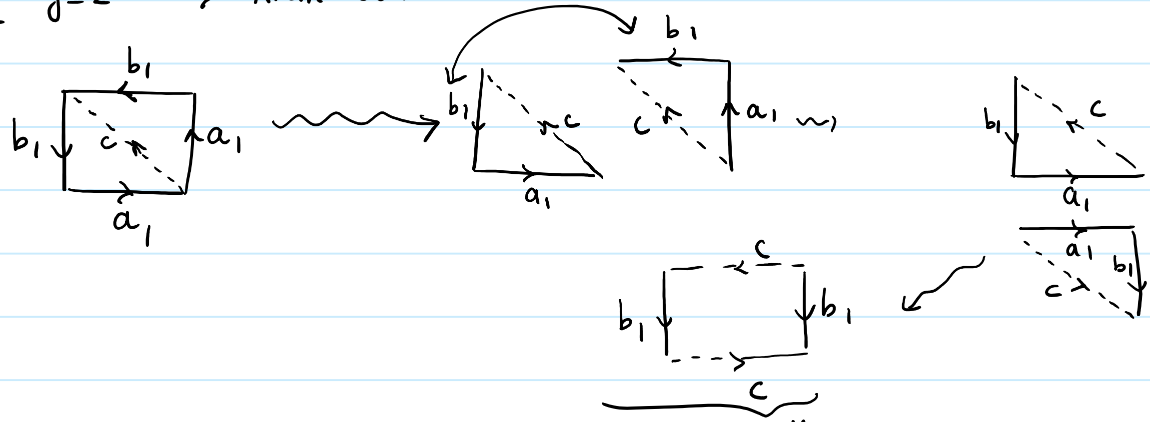
$\underline{\text{Ex}}$ $N_g = \underbrace{\mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{g\text{-Copies}} \rightsquigarrow$ Take a $2g$-gon and identify the edges as follows:

$g=3$



$$\rightsquigarrow \pi_1(N) = \frac{\underset{\langle a_1\rangle}{\mathbb{Z}} * \underset{\langle a_2\rangle}{\mathbb{Z}} \cdots * \underset{\langle a_g\rangle}{\mathbb{Z}}}{\langle a_1^2 a_2^2 \cdots a_g^2\rangle}$$

$\underline{\text{Special Case}}$ $g=2 \rightsquigarrow$ klein bottle



klein bottle

$\underline{\text{Cor}}$ If $g\neq h$, then $\pi_1(M_g) \not\approx \pi_1(M_h)$, abelianization of $\pi_1(M_g) = \overbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}^{g\text{-times}}$

$\qquad\qquad$ " $\qquad$ " $\pi_1(M_h) = \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{h\text{-times}}$

$\Rightarrow M_g \not\cong M_h$.

$\underline{\text{Ex}}$ Show that for $g\neq h$, $N_g \not\cong N_h$.

Prop (c) If Y is obtained from X by attaching n-cells for $n > 2$, then the inclusion $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X, x_0) \approx \pi_1(Y, x_0)$

Cor For a path connected cell complex X, the inclusion induces $\pi_1(X^2, x_0) \approx \pi_1(X, x_0)$
2-skeleton

Extra Example for Van Kampen:

$X = \mathbb{R}^3 \setminus A$ A:

First, take a 3-dim disk D^3 around A as:

$\mathbb{R}^3 \setminus A \xrightarrow[\text{deform retracts}]{} D^3 \setminus A$

Consider a torus T around A_2 such that as in
(T is disjoint from A_1 and intersects $\underset{\partial D^3}{S^2}$ in one pt.)

$\rightsquigarrow D^3 \setminus A \xrightarrow[\text{deform retracts}]{} S^2 \cup T \rightsquigarrow \pi_1(\mathbb{R}^3 \setminus A) \approx \pi_1(S^2) * \pi_1(T) \approx \mathbb{Z} \times \mathbb{Z}$