Lecture 8

Wednesday, February 15, 2017   9:51 AM

<ins>Recall</ins>: A <ins>Covering Space</ins> of a space X is a space $\tilde{X}$ with a map $p: \tilde{X} \longrightarrow X$ s.t.

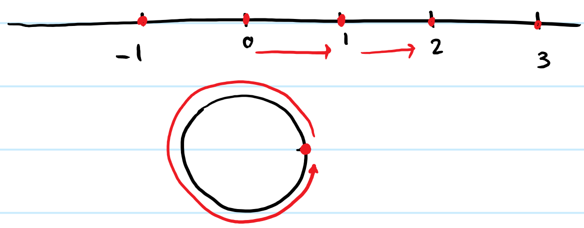
* for any $x \in X$ there exists an open nbd $x \in U \subset X$ s.t. $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$ each of which are mapped homeo onto U by p.

<ins>Def</ins> For any $x \in X$, $p^{-1}(x) \subset \tilde{X}$ is called a <ins>fiber</ins>.

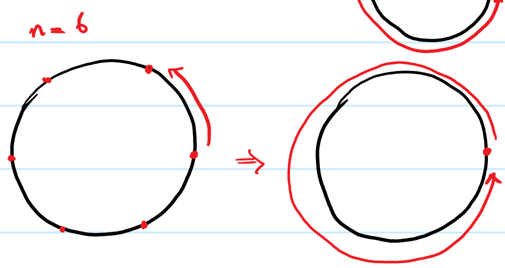
- Cardinality of $p^{-1}(x)$ is locally constant $\Rightarrow$ It's constant if X is connected.

<ins>Def</ins> $(\tilde{X}, p)$ is called an n-sheeted cover of X if for any $x \in X$, $p^{-1}(x)$ consists of n pts.

<ins>Ex</ins>
- $\mathbb{R}$, $p: \mathbb{R} \longrightarrow S^1$
  $t \longmapsto (\cos(2\pi t), \sin(2\pi t))$



- S^1, $p_n: S^1 \longrightarrow S^1$
  $z \longrightarrow z^n$



n-sheeted Covering

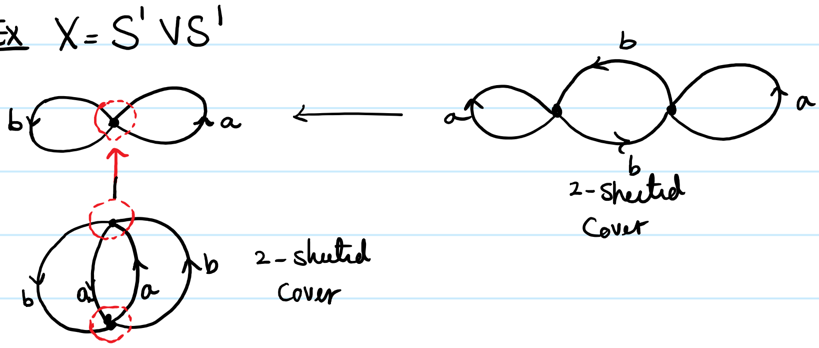
<ins>Def</ins>: Covering spaces $p_1: \tilde{X}_1 \longrightarrow X$ and $p_2: \tilde{X}_2 \longrightarrow X$ are called isomorphic if there exists a homeo. $f: \tilde{X}_1 \longrightarrow \tilde{X}_2$ such that $p_1 = p_2 f$.

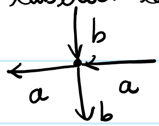
- if $\tilde{x}_0 \in p_1^{-1}(x_0)$ then $x_0 = p_1(\tilde{x}_0) = p_2 f(\tilde{x}_0) \rightsquigarrow f(\tilde{x}_0) \in p_2^{-1}(x_0)$

$\Rightarrow$ f maps a pt in fiber $p_1^{-1}(x_0)$ to a pt in fiber $p_2^{-1}(x_0)$.

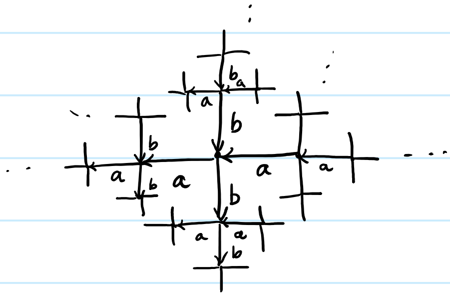
$\Rightarrow$ For $n \neq m$, (S^1, p_n) and (S^1, p_m) are not isomorphic.

<ins>Ex</ins> $X = S^1 \vee S^1$



In fact, any graph $\tilde{X}$ with deg 4 vertices, oriented and labeled edges with a and b such that near each vertex it looks like 
is a Covering Space of $S^1 \vee S^1$.

- $\tilde{X}$ Simply Connected if it has no loop.



Important: $p: \tilde{X} \longrightarrow X$, $\underset{\text{subgroup}}{p_*(\pi_1(\tilde{X},\tilde{x}_0)) < \pi_1(X,x_0)}$; $\tilde{x}_0 \in p^{-1}(x_0)$

Recall (Homotopy lifting property): Let $f_t: Y \longrightarrow X$ be a homotopy and $\tilde{f}_0$ be a lift of f_0 i.e. $p\tilde{f}_0 = f_0$. Then there exists a unique homotopy $\tilde{f}_t: Y \longrightarrow \tilde{X}$ of $\tilde{X}$ which lifts f_t.

① For any path $f: I \longrightarrow X$ starting at x_0 and any $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift $\tilde{f}: I \longrightarrow \tilde{X}$ starting at $\tilde{x}_0$.

② For any homotopy of paths $f_t: I \longrightarrow X$ starting at x_0 and any $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift $\tilde{f}_t: I \longrightarrow \tilde{X}$ starting at $\tilde{x}_0$ ($\tilde{f}_0$: unique lift of f_0 at $\tilde{x}_0$)

Remark If the paths γ_0 and γ_1 starting at x_0 are homotopic $\Rightarrow$ Their lifts $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ starting at $\tilde{x}_0$ are homotopic.

Def $\Phi_{\tilde{x}_0}: \pi_1(X,x_0) \longmapsto p^{-1}(x_0)$
$[g] \longmapsto \tilde{g}(1)$ lift of g starting at $\tilde{x}_0$.

Prop $p_*(\pi_1(\tilde{X},\tilde{x}_0)) < \pi_1(X,x_0)$ is equal to $\Phi_{\tilde{x}_0}^{-1}(\tilde{x}_0)$ i.e. Consists of homotopy classes of loops based at x_0 whose lift starting at $\tilde{x}_0$ is a loop.

Prop p_* is injective.

Pf Suppose $\tilde{f}_0: I \longrightarrow \tilde{X}$ be a loop at $\tilde{x}_0$ s.t. $p_*[\tilde{f}_0] = 0 \Rightarrow f_0 = p\tilde{f}_0$ is homo. trivial. Let $f_t: I \longrightarrow X$ be a homotopy btn f_0 and $f_1 = x_0$. Cor 2 implies that there exists a lift $\tilde{f}_t: I \longrightarrow \tilde{X}$ of f_t which is a homotopy of $\tilde{f}_0$ to $\tilde{f}_1$. $\tilde{f}_1$ is a lift of Constant loop which is Constant. $\Rightarrow \tilde{f}_0$ is homotopically trivial.

$\underset{\text{Covering Space}}{p: (\tilde{X},\tilde{x}_0) \longmapsto (X,x_0)} \Longrightarrow \underset{\text{subgroup}}{p_*(\pi_1(\tilde{X},\tilde{x}_0)) < \pi_1(X,x_0)}$

Lem Covering spaces $p_1: (\tilde{X}_1,\tilde{x}_1) \longrightarrow (X,x_0)$ and $p_2: (\tilde{X}_2,\tilde{x}_2) \longmapsto (X,x_0)$ are isomorphic by an iso. $f: \tilde{X}_1 \longrightarrow \tilde{X}_2$ s.t. $f(\tilde{x}_1) = \tilde{x}_2$. Then, $p_{1*}(\pi_1(\tilde{X}_1,\tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2,\tilde{x}_2))$

pf $p_2 f = p_1 \rightsquigarrow p_{2*} f_* = p_{1*}$, $f_* : \pi_1(\tilde{X}_1, \tilde{x}_1) \longrightarrow \pi_1(\tilde{X}_2, \tilde{x}_2)$ iso. $\Rightarrow \operatorname{im}(p_{1*}) = \operatorname{im}(p_{2*})$

Prop

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p\,:\ \text{Covering space}"] \\ (Y, y_0) \arrow[ur, dashed, "\overset{?}{\exists}\tilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}$$

① Suppose Y is path connected, and locally path connected. Then a lift $\tilde{f}$ exists iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

② If such a lift exists, then the lift is <u>unique</u>.

Cor Assume X is path connected. If for path connected covering spaces $p_1 : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (X, x_0)$ and $p_2 : (\tilde{X}_2, \tilde{x}_2) \longrightarrow (X, x_0)$ we have $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$ then covering spaces $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ are isom. via an isom. $f : \tilde{X}_1 \longrightarrow \tilde{X}_2$ such that $f(\tilde{x}_1) = \tilde{x}_2$.

Pf:

$$\begin{tikzcd} & (\tilde{X}_1, \tilde{x}_1) \arrow[d, "p_1"] \\ (\tilde{X}_2, \tilde{x}_2) \arrow[ur, dashed, "\tilde{p}_2"] \arrow[r, "p_2"'] & (X, x_0) \end{tikzcd}$$

$p_1 \tilde{p}_2 = p_2$

lift of p_2

$$\begin{tikzcd} & (\tilde{X}_2, \tilde{x}_2) \arrow[d, "p_2"] \\ (\tilde{X}_1, \tilde{x}_1) \arrow[ur, dashed, "\tilde{p}_1"] \arrow[r, "p_1"'] & (X, x_0) \end{tikzcd}$$

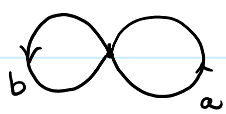
$p_2 \tilde{p}_1 = p_1$

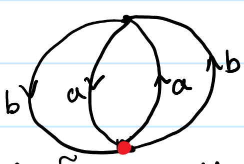
$p_1(\underbrace{\tilde{p}_2 \tilde{p}_1}_{\text{lifting of } p_1}) = p_2 \tilde{p}_1 = p_1$

$$\begin{tikzcd} & (\tilde{X}_1, \tilde{x}_1) \arrow[d, "p_1"] \\ (\tilde{X}_1, \tilde{x}_1) \arrow[ur, dashed, "\mathbb{1}"] \arrow[r, "p_1"'] & (X, x_0) \end{tikzcd}$$

$\xrightarrow{\text{unique}} \tilde{p}_2 \tilde{p}_1 = \mathbb{1}$, similarly $\tilde{p}_1 \tilde{p}_2 = \mathbb{1} \Rightarrow \tilde{p}_1, \tilde{p}_2$ isomorphism.

Ex $(S^1, p_n) \Rightarrow \langle n \rangle$: subgroup gen. by n.





$\langle ba^{-1}, a^2, ab \rangle$

Prop Suppose X and $\tilde{X}$ are path connected. The number of sheets for $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ equals with the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.

Pf Let $H = p_*(\pi_1(\tilde{X}, x_0))$ Φ : Coset of $H \longmapsto p^{-1}(x_0)$

$H[g] \longmapsto \Phi_{\tilde{x}_0}([g])$

Injective: if $\Phi(H[g_1]) = \Phi(H[g_2]) \Rightarrow \tilde{g}_1(1) = \tilde{g}_2(1) \Rightarrow \tilde{g}_1 \cdot \overline{\tilde{g}_2}$: loop based at $\tilde{x}_0$

$p_*([\tilde{g}_1 \cdot \overline{\tilde{g}_2}]) = [g_1][g_2]^{-1} \in H \Rightarrow H[g_1] = H[g_2]$.

Surjective Let $\tilde{x}_1 \in p^{-1}(x_0)$. Since $\tilde{X}$ is path connected, we may connect $\tilde{x}_0$ to $\tilde{x}_1$ by

Surjective Let $\tilde{x}_1 \in P^{-1}(x_0)$. Since X is path connected, we may connect $\tilde{x}_0$ to $\tilde{x}_1$ by a path $\tilde{g}$ in $\tilde{X}$. Let $g = p\tilde{g}$. Then $\Phi(H[g]) = \tilde{x}_1$.

(g: loop bound at x_0)

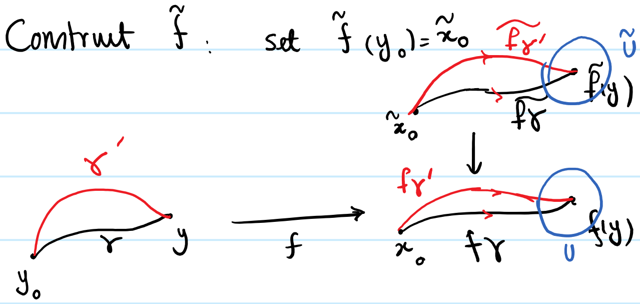
Proof of lifting property

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

(Y: path connected)

$p\tilde{f} = f \Rightarrow p_* \tilde{f}_* = f_* \Rightarrow \operatorname{im} f_* = \operatorname{im}(p_* \tilde{f}_*) \subset \operatorname{im}(p_*)$ ✓

Construct $\tilde{f}$: set $\tilde{f}(y_0) = \tilde{x}_0$



$\Rightarrow \tilde{f}(y) = \widetilde{f\gamma}(1)$

- Well-defined : $\gamma \cdot \bar{\gamma}'$: loop bound at y_0. $f_*([\gamma \cdot \bar{\gamma}']) = [f\gamma \cdot \overline{f\gamma'}] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

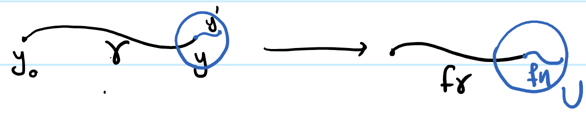
$\Rightarrow$ lift of $f\gamma \cdot \overline{f\gamma'}$ starting at $\tilde{x}_0$ is a loop. $\widetilde{f\gamma} \cdot \widetilde{\overline{f\gamma'}}$: loop

$\Rightarrow \widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1)$

- Conti Let U be an open nbd of $f(y)$, such that it has a lift $\tilde{U} \ni \tilde{f}(y)$ for which $p: \tilde{U} \to U$ is a homo.

Y locally path connected, pick a path conn. open nbd V of y s.t. $f(V) \subset U$. For any $y' \in V$, pick a path η from y to y' in V.

$\Rightarrow f\eta \subset U$



$\Rightarrow$ lift $f\eta$ starting at $\tilde{f}(y)$ is equal to $p^{-1}f\eta$ for $p: \tilde{U} \to U$

$\Rightarrow \tilde{f}(y') \in \tilde{U} \Rightarrow \tilde{f}(V) \subseteq \tilde{U}$

Uniqueness : Let $\tilde{f}_1, \tilde{f}_2 : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ be lifts of f. We show that the set of pts $y \in Y$ for which $\tilde{f}_1(y) = \tilde{f}_2(y)$ is both open and closed.

. $\tilde{f}_1(y) = \tilde{f}_2(y)$, let N be a nbd of y s.t. $\tilde{f}_1(N), \tilde{f}_2(N) \subset \tilde{U}$

$f = p\tilde{f}_1 = p\tilde{f}_2$ and $p: \tilde{U} \to U$ is homo $\rightsquigarrow \tilde{f}_1 = \tilde{f}_2$ on N.

. If $\tilde{f}_1(y) \neq \tilde{f}_2(y)$, there exist a nbd N of y s.t. $\tilde{f}_1(N) \subset \tilde{U}_1$ and $\tilde{f}_2(N) \subset \tilde{U}_2$ s.t. $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$ and p maps $\tilde{U}_1$ and $\tilde{U}_2$ homeo onto U.

$\Rightarrow \tilde{f}_1 \neq \tilde{f}_2$ on N.

$\Longrightarrow$ Since Y is connected, $\tilde{f}_1 = \tilde{f}_2$.

(We don't need path connectedness or locally path connectedness for uniqueness)