Lecture 9

Sunday, February 19, 2017 5:54 PM

Recall: Covering space $\Longrightarrow$ subgroup

$p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ $\quad$ $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) < \pi_1(X, x_0)$

invariant under isom.

Prop Suppose X is path connected and locally path connected. Then any two path connected covering space $p_1: (\tilde{X}_1, \tilde{x}_1) \longrightarrow (X, x_0)$ with $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$

$p_2: (\tilde{X}_2, \tilde{x}_2) \longrightarrow (X, x_0)$

are iso via an isom. $f: \tilde{X}_1 \longrightarrow \tilde{X}_2$ s.t. $f(\tilde{x}_1) = \tilde{x}_2$.

Prop (Lifting property)

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p: \text{Covering space}"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}$$

path connected

locally path connected

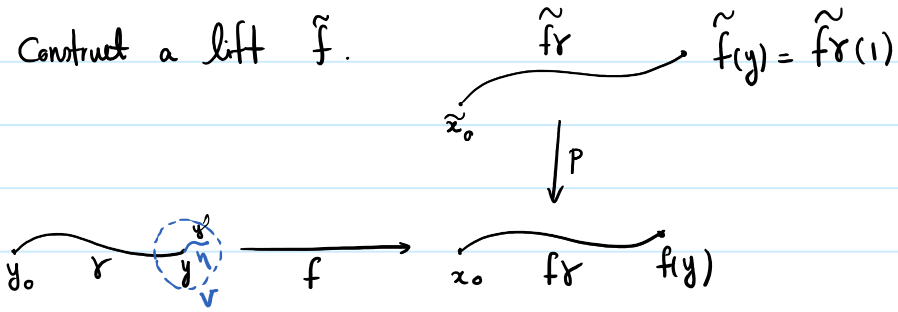
① a lift $\tilde{f}$ exists $\Longleftrightarrow$ $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, x_0))$

② lift $\tilde{f}$ is unique.

pf ($\Longrightarrow$) ✓

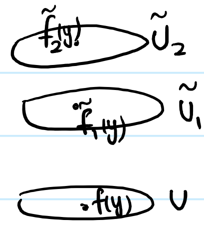
($\Longleftarrow$) To prove we construct a lift $\tilde{f}$.

$\tilde{f}(y_0) = \tilde{x}_0$.



- well-defined: ✓
- Continuous: Let $\tilde{U}$ be an open nbd of $\tilde{f}(y)$. We need to show that there exists an open nbd V of y s.t. $\tilde{f}(V) \subset \tilde{U}$. Let U' be an evenly covered nbd of $f(y)$ and $\tilde{U}'$ be a lift of U' containing $\tilde{f}(y)$ s.t. $p: \tilde{U}' \longrightarrow U'$ is a homeo.
  Then, $\tilde{U} \cap \tilde{U}'$ is an open nbd of $\tilde{f}(y)$ and $p|_{\tilde{U} \cap \tilde{U}'}$ is a homeo. f is conti.
  $\Rightarrow$ let V be a path connected open nbd of y s.t. $f(V) \subset p(\tilde{U} \cap \tilde{U}')$.
  For any $y' \in V$, take a path η from y to y'. Then $\tilde{f}(y') = \widetilde{f(\gamma \cdot \eta)}(1) = \tilde{f\eta}(1)$
  $f\eta \subset \tilde{U} \cap \tilde{U}'$ $\rightsquigarrow$ lift of $f\eta$ starting at $\tilde{f}(y)$ lies in $\tilde{U} \cap \tilde{U}'$ $\Rightarrow$ $\tilde{f}(y') \in \tilde{U} \cap \tilde{U}'$
  $\Rightarrow \tilde{f}(V) \subset \tilde{U}$.
- Uniqueness: Let $\tilde{f}_1$ and $\tilde{f}_2$ be lifts of f. Suppose $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. Then take an evenly covered nbd U of $f(y)$ and suppose $\tilde{f}_1(y) \in \tilde{U}_1$ and $\tilde{f}_2(y) \in \tilde{U}_2$ be lifts

of U such that $p:\tilde U_1 \to U$ and $p:\tilde U_2 \to U$ is a homeo.
$\tilde f_1$ and $\tilde f_2$ are conti $\Rightarrow$ there is a nbd V of y s.t. $\tilde f_1(V) \subset \tilde U_1$ and $\tilde f_2(V) \subset \tilde U_2$ $\Rightarrow$ $\tilde f_1(y') \neq \tilde f_2(y')$ for any $y' \in V \Rightarrow$ The set of pts where $\tilde f_1(y) \neq \tilde f_2(y)$ is open.

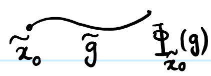


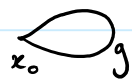
Suppose $y \in Y$ be a pt s.t. $\tilde f_1(y) = \tilde f_2(y)$. Take an evenly covered nbd U of $f(y)$ and a lift $\tilde U$ of U containing $\tilde f_1(y) = \tilde f_2(y)$ as above. Then for a nbd V of y we have $\tilde f_1(V), \tilde f_2(V) \subset \tilde U$. $p\tilde f_1 = p\tilde f_2$ and $p:\tilde U \to U$ is a homeo. $\Rightarrow$ p is injective on $\tilde U$.
$\Rightarrow \tilde f_1 = \tilde f_2$ on V. The set of pts where $\tilde f_1 = \tilde f_2$ is open.
$\Rightarrow$ Y is connected and $\tilde f_1(y_0) = \tilde f_2(y_0) = \tilde x_0$ implies that $\tilde f_1 = \tilde f_2$.

<u>Prop</u> $p:(\tilde X, \tilde x_0) \to (X, x_0)$ is a path connected covering space of a path connected space X. The cardinality of the fiber $p^{-1}(x_0)$ is equal to the index of $p_*(\pi_1(\tilde X, \tilde x_0))$ in $\pi_1(X, x_0)$.

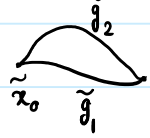
<u>Pf</u> Let $H = p_*(\pi_1(\tilde X, \tilde x_0))$. Let Φ : Cosets of $H \longmapsto p^{-1}(x_0)$

$$H[g] \longmapsto \Phi_{\tilde x_0}([g]) = \tilde g(1)$$

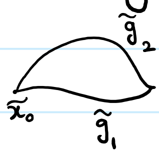




- Well-defined : $H[g_1] = H[g_2] \Rightarrow [g_1][g_2^{-1}] \in H$
 $\Rightarrow [g_1 g_2^{-1}] \in H \Rightarrow \widetilde{g_1 g_2^{-1}}$ is a loop.
 $\Rightarrow \tilde g_1 \cdot \tilde{\bar g}_2(1) = \tilde x_0$.
 $\Rightarrow \tilde g_1(1) = \tilde g_2(1)$ ✓



- Surjective : $\tilde X$ is path connected. For $\tilde x \in p^{-1}(x_0)$ take a path γ connecting $\tilde x_0$ to $\tilde x$ $\Rightarrow$ Let $g = p(\gamma) \Rightarrow \Phi(H[g]) = \tilde x$.
- Injective : Suppose $\Phi(H[g_1]) = \Phi(H[g_2]) \Rightarrow \tilde g_1(1) = \tilde g_2(1)$.
 $\Rightarrow \tilde g_1 \cdot \bar{\tilde g}_2$ is the lift of $g_1 \cdot \bar g_2$. $\Rightarrow [g_1 \cdot \bar g_2] \in H$
 $\Rightarrow [g_1][g_2]^{-1} \in H \Rightarrow H[g_1] = H[g_2]$.



<u>Q</u> : Suppose X is path connected and locally path connected. Let $H < \pi_1(X, x_0)$. Is there a path-conn. covering space $p:(\tilde X, \tilde x_0) \to (X, x_0)$ s.t. $p_* \pi_1(\tilde X, \tilde x_0) = H$? $\boxed{\text{NO}}$
- X should be <u>semilocally simply connected</u> !

<u>Def</u> A top. space X is semi locally simply connected if for any $x \in X$ there exists a nbd U of x such that $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Why this condition is necessary ? Let H be the trivial subgroup. Then $p_* \pi_1(\tilde X, \tilde x_0) = H$.
$\Rightarrow \pi_1(\tilde X, \tilde x_0)$ is trivial $\Rightarrow \tilde X$ is simply connected. Take an evenly covered nbd U of x and let $\tilde U$ be a lift of U s.t. $p:\tilde U \to U$ is a homeo. For any $[f] \in \pi_1(U, x)$, $p^{-1}f \subset \tilde U$

is homotopically trivial in $\tilde{X}$. Compose null homotopy with p. Shows f is null hom. in $X \Rightarrow \pi_1(U,x) \longrightarrow \pi(X,x)$ is trivial.

Prop: Suppose X is path connected, locally path connected and semilocally simply connected. Then for any subgroup $H < \pi_1(X,x_0)$ there is a path connected covering space $p:(\tilde{X}_H, \tilde{x}_0) \longrightarrow (X,x_0)$ such that $p_*(\pi_1(\tilde{X}_H,\tilde{x}_0)) = H$.

Def The simply connected covering space $p: \tilde{X} \longrightarrow X$ is called the <u>universal cover</u> of X.

why? $\tilde{X}$ is a covering space for every other covering space of X.

Ex $p: \mathbb{R} \longrightarrow S^1$ $\rightsquigarrow$ $\mathbb{R}$ is the universal cover.

$t \longmapsto (\cos(2\pi t), \sin(2\pi t))$

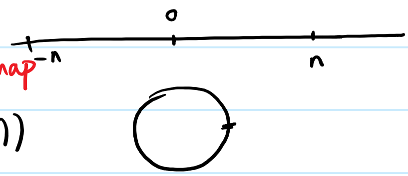
$P_n: S^1 \longrightarrow S^1$

$z \longrightarrow z^n$

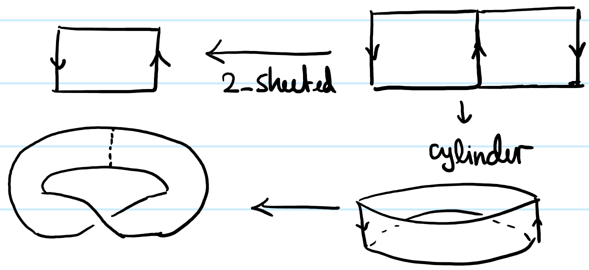
$\Rightarrow$

$$\begin{array}{ll} \mathbb{R} & t \\ \downarrow & \downarrow \\ S^1 & (\cos(2\pi \frac{t}{n}), \sin(2\pi \frac{t}{n})) \\ \downarrow & \downarrow P_n \\ S^1 & (\cos(2\pi t), \sin(2\pi t)) \end{array}$$

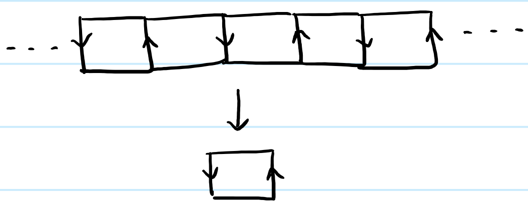
$\rightsquigarrow$ Covering map



Ex X: Mobius band

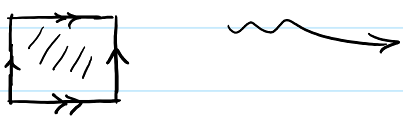


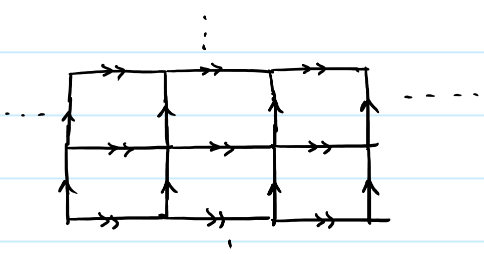
Universal Cover:



Ex X: Torus

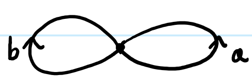
Universal Cover: $\mathbb{R}^2$

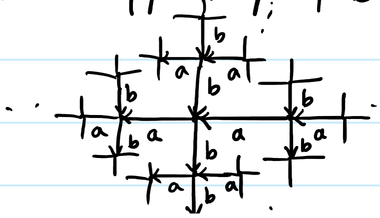




Similarly: $\mathbb{R}^2$ with an app. map p is the universal cover of the Klein bottle.

Ex





$\Leftarrow$ Universal Cover

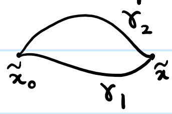
Ex $\mathbb{R}P^n \cong S^n/_{(x\sim -x)} \Rightarrow q: S^n \longrightarrow \mathbb{R}P^n$ is a Covering Space. For $n\geq 2$, S^n is simply Connected $\Rightarrow (S^n, q)$ is the universal cover of $\mathbb{R}P^n$ for $n\geq 2$.

($\mathbb{R}P^1 \cong S^1 \Rightarrow$ it's universal Cover is $\mathbb{R}$.)

Idea of proof for H = trivial subgroup :

Suppose $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a simply Connected covering space.

- $\tilde{X}$ is simply Connected $\Rightarrow$ for any $\tilde{x} \in \tilde{X}$ there is a unique homotopy class of path connecting $\tilde{x}_0$ to $\tilde{x}$.



$\gamma_1\overline{\gamma}_2$ is null homotopic.

$\gamma_1\overline{\gamma}_2 \simeq c_{\tilde{x}_0} \Rightarrow \gamma_1\overline{\gamma}_2\gamma_2 \simeq c\cdot\gamma_2$

$\Rightarrow$ any $\tilde{x} \in \tilde{X} \Longleftrightarrow$ A homotopy class of paths Starting at $\tilde{x}_0$ $\quad \Rightarrow \gamma_1 \simeq \gamma_2$.

$\underset{HLP}{\Longleftrightarrow}$ A homotopy class of paths starting at x_0.

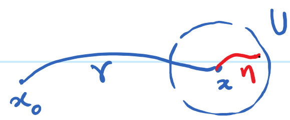
$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\} \qquad \begin{aligned} p: \tilde{X} &\longrightarrow X \\ [\gamma] &\longrightarrow \gamma(1) \end{aligned}$$

Basis for topology : $\mathcal{U} = \{U \subset X \mid U \text{ is path connected, open }, \pi_1(U) \xrightarrow[\text{trivial}]{} \pi_1(X)\}$

(Ex: $\mathcal{U}$ is a basis of topology for X)

For any $U \in \mathcal{U}$, and a path γ from x_0 to a pt in U

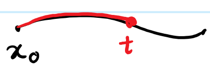
$$U_{[\gamma]} = \{[\gamma\cdot\eta] \mid \eta \underset{\text{path}}{\subset} U \text{ s.t. } \eta(0) = \gamma(1)\}$$



- $\{U_{[\gamma]}\}$ form a basis for topology on $\tilde{X}$
- $p: \tilde{X} \longrightarrow X$ is a Covering Space. $\left(\begin{aligned} p: U_{[\gamma]} &\longrightarrow U \\ [\gamma\cdot\eta] &\longrightarrow \gamma\cdot\eta(1) \end{aligned} \text{ is bijective.} \right)$

* $\tilde{X}$ is simply Connected.

① path Connected : $[\gamma] \in \tilde{X}$



$$\gamma_t(s) = \begin{cases} \gamma(s) & 0 \leq s \leq t \\ \gamma(t) & 0 \leq t \leq 1 \end{cases}$$

$t \longmapsto [\gamma_t]$: path in $\tilde{X}$ Connecting $[\gamma_0] = [x_0]$ to $[\gamma_1] = [\gamma]$.

$\rightsquigarrow$ path Connected.

It's infact the lift of γ starting at $[x_0]$, because $p[\gamma_t] = \gamma(t)$.

$\rightsquigarrow$ For any $[\gamma] \in \pi_1(X, x_0)$ lift of γ starting at $[x_0]$ is a path Connecting $[x_0]$ to $[\gamma]$. Thus if γ is null homotopic $\rightsquigarrow$ lift of γ is not a loop.

$\Rightarrow$ im(p_*) is the trivial subgroup of $\pi_1(X, x_0)$.