Homework 9

Due: Apr 25

1. Let V be a vector space over $\mathbb{R}$ and $B : V \otimes V \to \mathbb{R}$ a bilinear form. Suppose that there is a half-dimensional subspace L of V so that $B(v, w) = 0$ for all $v, w \in L$. Prove that the signature of B is zero.

2. Show that if X is closed, oriented, simply connected n-manifold with $H_i(X) = 0$ for $0 < i < n$, then it is homotopy equivalent to S^n.

3. Let M be a smooth, closed, oriented n-manifold and $f : M \to S^{n-4i}$ be a smooth map. Show that for every regular value y of f,

$$\langle L_i(M) \cup f^\star(u), \mu_M \rangle = \sigma(f^{-1}(y)),$$

where u and μ_M denote the fundamental cohomology class of S^{n-4i} and fundamental homology class of M.

4. a. Let ξ be the underlying oriented 4-plane bundle of the quaternion line bundle over $\mathbb{HP}^m$. Use the Gysin sequence for ξ to compute $H^\star(\mathbb{HP}^m)$.

   b. For $m = 1$ i.e. $\mathbb{HP}^1 \cong S^4$, show that

$$p_1(\xi) = -2u, \quad \textit{and} \quad e(\xi) = u,$$

   where u is the generator of $H^4(\mathbb{HP}^m)$.

   c. Milnor-Stasheff 20-A.

5. Milnor-Stasheff 19-A.

6. a. Suppose X be a smooth, closed 4-manifold, and J be an almost complex structure on X, i.e. J makes tangent space of X into a complex vector bundle. Show that, considering J, $c_1^2[X] = 3\sigma(X) + 2\chi(X)$.

   b. Prove that S^4 do not admit any almost-complex structure.

7. Fix a closed, smooth, oriented manifold M^7 with $H^3(M) = H^4(M) = 0$. Let B_1 and B_2 be oriented 8-manifolds with boundary $\partial B_i = M$. Let $C = B_1 \cup_M (-B_2)$. Let $i : H^4(B_j, M) \to H^4(B_j)$ be the map from the long exact sequence of a pair. Define $\sigma(B_j)$ to be the signature of the bilinear form

$$H^4(B_j, M; \mathbb{Q}) \otimes H^4(B_j, M; \mathbb{Q}) \to \mathbb{Q}$$
$$(a, b) \mapsto \langle a \cup b, [B_j] \rangle$$

Prove that

$$\langle p_1(TC)^2, [C] \rangle = \langle i^{-1} p_1(TB_1)^2, [B_1] \rangle - \langle i^{-1} p_1(TB_2)^2, [B_2] \rangle$$
$$\sigma(C) = \sigma(B_1) - \sigma(B_2)$$