Introduction to spectral sequences

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Abstract

The words "spectral sequence" strike fear into the hearts of many hardened mathematicians. These notes will attempt to demonstrate that spectral sequences are not so scary, and also very powerful.

This is an unfinished handout for my algebraic topology class. In particular I did not have time to reproduce here the little spectral sequence diagrams showing where all the arrows go.

For a comprehensive introduction to spectral sequences, see [3]. For more nice explanations of spectral sequences, see [1] and [2]. Finally, the original paper [4] is a good read.

A short exact sequence of chain complexes gives rise to a long exact sequence in homology, which is a fundamental tool for computing homology in a number of situations. There is a natural generalization of a short exact sequence of chain complexes, called a "filtered chain complex". Associated to a chain complex with a filtration is an algebraic gadget generalizing the long exact sequence, which is called a spectral sequence, and which can help compute the homology of the chain complex.

1 The long exact sequence in homology

We begin by reviewing the long exact sequence in homology associated to a short exact sequence of chain complexes, from a point of view which naturally generalizes to spectral sequences. Consider a chain complex C_* with a subcomplex F_0C_*. We now have a short exact sequence of chain complexes

$$0 \longrightarrow F_0C_* \longrightarrow C_* \longrightarrow C_*/F_0C_* \longrightarrow 0.$$

A fundamental lemma in homological algebra asserts that there is then a long exact sequence in homology

$$\cdots \longrightarrow H_i(F_0C_*) \longrightarrow H_i(C_*) \longrightarrow H_i(C_*/F_0C_*) \xrightarrow{\delta} H_{i-1}(F_0C_*) \longrightarrow \cdots .$$

The connecting homomorphism δ is defined as follows: given a homology class $\alpha \in H_i(C_*/F_0C_*)$, choose $x \in C_i$ representing it; then $\partial x \in F_0C_{i-1}$, and we define $\delta(\alpha) = [\partial x]$.

Now suppose that our goal is to compute the homology of the whole complex C_*, and that it is somehow easier to compute the homologies of the subcomplex F_0C_* and the quotient complex C_*/F_0C_*. The long exact sequence can be broken into a short exact sequence

$$0 \to \operatorname{Coker}\left(\delta|_{H_{i+1}(C_*/F_0C_*)}\right) \to H_i(C_*) \to \operatorname{Ker}\left(\delta|_{H_i(C_*/F_0C_*)}\right) \to 0.$$

If we leave the index i implicit, then we can write this more concisely as

$$0 \longrightarrow \operatorname{Coker}(\delta) \longrightarrow H_*(C_*) \longrightarrow \operatorname{Ker}(\delta) \longrightarrow 0.$$

In conclusion, the procedure for computing $H_*(C_*)$ is the following:

1. Compute $H_*(F_0C_*)$ and $H_*(C_*/F_0C_*)$.

2. Consider the two-term chain complex

$$H_*(C_*/F_0C_*) \xrightarrow{\delta} H_*(F_0C_*).$$

   Denote its homology groups by G_1H_* and G_0H_*.

3. There is now a short exact sequence

$$0 \longrightarrow G_0H_* \longrightarrow H_*(C_*) \longrightarrow G_1H_* \longrightarrow 0.$$

   Modulo the problem of extensions, this determines $H_*(C_*)$.

2 Filtrations

A *filtered R-module* is an R-module A with an increasing sequence of submodules $F_pA \subset F_{p+1}A$ indexed by $p \in \mathbb{Z}$, such that $\bigcup_p F_pA = A$ and $\bigcap_p F_pA = \{0\}$. The filtration is *bounded* if $F_pA = \{0\}$ for p sufficiently small and $F_pA = A$ for p sufficiently large.

The *associated graded* module is defined by $G_pA = F_pA/F_{p-1}A$. In favorable cases, this inductively determines A by means of the short exact sequences

$$0 \longrightarrow F_{p-1}A \longrightarrow F_pA \longrightarrow G_pA \longrightarrow 0.$$

To give an example having nothing to do with the rest of these notes, let A be the $\mathbb{R}$-module of differentiable functions $f : \mathbb{R} \to \mathbb{R}$, and let F_pA be the submodule of functions such that the $(p+1)^{st}$ derivative at 0 vanishes: $f^{(p+1)}(0) = 0$. There is then an isomorphism $G_pA \xrightarrow{\simeq} \mathbb{R}$ sending $f \longmapsto f^{(p)}(0)$.

A *filtered chain complex* is a chain complex (C_*, ∂) together with a filtration $\{F_pC_i\}$ of each C_i, such that the differential preserves the filtration, namely $\partial(F_pC_i) \subset F_pC_{i-1}$.

Note that ∂ induces a well-defined differential $\partial : G_pC_i \to G_pC_{i-1}$. We thus have an associated graded chain complex G_pC_*.

The filtration on C_* also induces a filtration on the homology of C_* defined by

$$F_pH_i(C_*) = \{\alpha \in H_i(C_*) \mid (\exists x \in F_pC_i)\ \alpha = [x].\}.$$

This has associated graded pieces $G_pH_i(C_*)$, which in favorable cases determine $H_i(C_*)$.

Now suppose that our goal is to compute the homology of C_*, and that it is somehow easier to compute the homology of the assocated graded chain complexes G_pC_*. Does $H_*(G_pC_*)$ determine $G_pH_*(C_*)$? We saw in the previous section that if the filtration has only one nontrivial term, i.e. if $F_{-1}C_* = \{0\}$ and $F_1C_* = C_*$, then $G_pH_*(C_*)$ is the homology of the two-term chain complex

$$H_*(G_1C_*) \xrightarrow{\delta} H_*(G_0C_*).$$

When the filtration has more nontrivial terms, the homology of C_* can be computed by "successive approximations", as we now explain.

3 Computing the homology of a filtered chain complex

Let (F_pC_*, ∂) be a filtered chain complex. Let us denote the associated graded module by

$$E^0_{p,q} = G_pC_{p+q} = F_pC_{p+q}/F_{p-1}C_{p+q}.$$

As remarked previously, the differential ∂ induces a differential on the associated graded module, which we now denote by

$$\partial_0 : E^0_{p,q} \longrightarrow E^0_{p,q-1}.$$

We denote the homology of the associated graded by

$$E^1_{p,q} = H_{p+q}(G_pC_*).$$

This can be regarded as a "first-order approximation" to the homology of C_*.

To get a "second-order approximation" to the homology of C_*, we define

$$\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$$

as follows. A homology class $\alpha \in E^1_{p,q}$ can be represented by a chain $x \in F_pC_{p+q}$ such that $\partial x \in F_{p-1}C_{p+q-1}$. We now define $\partial_1(\alpha) = [\partial x]$. It follows easily from $\partial^2 = 0$ that ∂_1 is well-defined and $\partial_1^2 = 0$. We now consider the homology

$$E^2_{p,q} = \frac{\mathrm{Ker}(\partial_1 : E^1_{p,q} \to E^1_{p-1,q})}{\mathrm{Im}(\partial_1 : E^1_{p+1,q} \to E^1_{p,q})}.$$

We saw previously that if the fitration has only one nontrivial term, then $E^2_{p,q} = G_pH_{p+q}(C_*)$. If the filtration has more nontrivial terms, then this might not be true.

In general, for every nonnegative integer r, we define an "r^{th}-order approximation" to $G_pH_{p+q}(C_*)$ by

$$E^r_{p,q} = \frac{\{x \in F_pC_{p+q} \mid \partial x \in F_{p-r}C_{p+q-1}\}}{F_{p-1}C_{p+q} + \partial(F_{p+r-1}C_{p+q+1})}. \tag{1}$$

Here the notation indicates the quotient of the numerator by its intersection with the denominator. So by contrast with the definition of homology, instead of considering cycles, we just consider chains in F_p whose differential "vanishes to order r", i.e. lives in F_{p-r}; and instead of modding out by the entire image of ∂, we only mod out by $\partial(F_{p+r-1})$.

Lemma 3.1 *Let (F_pC_*, ∂) be a filtered complex, and define $E^r_{p,q}$ by equation* (1). *Then:*

(a) ∂ induces a well-defined map

$$\partial_r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$$

satisfying $\partial_r^2 = 0$.

(b) E^{r+1} is the homology of the chain complex (E^r, ∂_r), i.e.

$$E^{r+1}_{p,q} = \frac{\mathrm{Ker}(\partial_r : E^r_{p,q} \to E^1_{p-r,q+r-1})}{\mathrm{Im}(\partial_r : E^1_{p+r,q-r+1} \to E^1_{p,q})}.$$

(c) $E^1_{p,q} = H_{p+q}(G_pC_*)$.

(d) *If the filtration of C_i is bounded for each i, then for every p, q, if r is sufficiently large then*

$$E^r_{p,q} = G_p H_{p+q}(C_*).$$

Proof. As with the fact that a short exact sequence of chain complexes induces a long exact sequence on homology, which this lemma generalizes, the proof is a straightforward but notationally messy exercise which you will only understand if you do it yourself. (The hard part of this lemma is not the proof, but rather finding the right statement.) □

Example 3.2 [algebraic example where you have to compute ∂_2 by zig-zagging.]

Example 3.3 Let us re-prove that the singular homology of a CW complex X agrees with the cellular homology. Let $C_*(X)$ denote the singular chain complex of X. Define a filtration on $C_*(X)$ by $F_p C_*(X) = C_*(X^p)$, where X^p denotes the p-skeleton of X. The associated graded is

$$E^0_{p,q} = C_{p+q}(X^p)/C_{p+q}(X^{p-1}).$$

The homology of this is, by definition, the relative homology

$$E^1_{p,q} = H_{p+q}(X^p, X^{p-1}).$$

Now recall that

$$H_{p+q}(X^p, X^{p-1}) \simeq \begin{cases} C^{\mathrm{cell}}_p(X), & q = 0, \\ 0, & q \neq 0, \end{cases}$$

where $C^{\mathrm{cell}}_p(X)$ is a free $\mathbb{Z}$-module with one generator for each p-cell. Furthermore there is a differential $\partial : C^{\mathrm{cell}}_p(X) \to C^{\mathrm{cell}}_{p-1}(X)$, which is the map

$$\partial : H_p(X^p, X^{p-1}) \longrightarrow H_{p-1}(X^{p-1}, X^{p-2})$$

induced by the long exact sequence of the triple (X^p, X^{p-1}, X^{p-2}), and which can be explicitly computed in various ways. It is easy to see from the definitions that this agrees with the map

$$\partial_1 : E^1_{p,0} \longrightarrow E^1_{p-1,0}.$$

Therefore E^2 is given in terms of the cellular homology by

$$E^2_{p,q} = \begin{cases} H_p^{\text{cell}}(X), & q = 0, \\ 0, & q \neq 0. \end{cases}$$

Now the key observation is that since the E^2 term is all supported in the row $q = 0$, the higher differentials ∂_r for $r \geq 2$ all necessarily vanish, since for each such arrow, either the domain or the range is zero. Hence $E^r_{p,q} = E^2_{p,q}$ for all $r \geq 2$. If X is finite-dimensional, so that the filtration is bounded, then it follows that $H_p(X) = H_p^{\text{cell}}(X)$. One can drop the finite-dimensionality assumption in various ways, e.g. by taking a direct limit.

To summarize, we make the following definition.

Definition 3.4 A *spectral sequence* consists of:

- An R-module $E^r_{p,q}$ defined for each $p, q \in \mathbb{Z}$ and each integer $r \geq r_0$, where r_0 is some nonnegative integer.
- Differentials $\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $\partial_r^2 = 0$ and E^{r+1} is the homology of (E^r, ∂_r), i.e.

$$E^{r+1}_{p,q} = \frac{\operatorname{Ker}(\partial_r : E^r_{p,q} \to E^1_{p-r,q+r-1})}{\operatorname{Im}(\partial_r : E^1_{p+r,q-r+1} \to E^1_{p,q})}.$$

A spectral sequence *converges* if for every p, q, if r is sufficiently large then ∂_r vanishes on $E^r_{p,q}$ and $E^r_{p+r,q-r+1}$. In this case, for each p, q, the module $E^r_{p,q}$ is independent of r for r sufficiently large, and we denote this by $E^\infty_{p,q}$.

For a given r, the collection of R-modules $\{E^r_{p,q}\}$, together with the differential ∂_r between them, is called the "E^r term" or the "r^{th} page" of the spectral sequence. One typically draws this as a chart where p is the horizontal coordinate and q is the vertical coordinate. One can regard a spectral sequence as a book, with pages indexed by r, in which each page is the homology of the previous page.

In terms of this definition, we have shown:

Proposition 3.5 *Let (F_pC_*, ∂) be a filtered complex. Then there is a spectral sequence $(E^r_{p,q}, \partial_r)$, defined for $r \geq 0$, with*

$$E^1_{p,q} = H_{p+q}(G_pC_*).$$

If the filtration of C_i is bounded for each i, then the spectral sequence converges to

$$E^\infty_{p,q} = G_pH_{p+q}(C_*).$$

[maybe explain how to compare spectral sequences]

Example 3.6 Let us re-compute the homology of the tensor product of two chain complexes (C_*, ∂) and (C'_*, ∂') over a field K. Assume $C_i = C'_i = 0$ for $i < 0$. Recall that the tensor product chain complex is defined by

$$(C \otimes C')_k = \bigoplus_{i+j=k} C_i \otimes C_j$$

and the differential is defined, for $\alpha \in C_i$ and $\beta \in C'_j$, by

$$\partial(\alpha \otimes \beta) = (\partial\alpha) \otimes \beta + (-1)^i \alpha \otimes (\partial'\beta).$$

To compute the homology of $C \otimes C'$, define a filtration on it by

$$F_p(C \otimes C')_k = \bigoplus_{i \leq p} C_i \otimes C_{k-i}.$$

The associated graded is then

$$E^0_{p,q} = G_p(C \otimes C')_{p+q} = C_p \otimes C'_q.$$

The differential on this is $\partial_0 = (-1)^p \otimes \partial'$. So by the universal coefficient theorem,

$$E^1_{p,q} = C_p \otimes H_q(C'_*).$$

Furthermore $\partial_1 = \partial \otimes 1$, so, since we are working over a field,

$$E^2_{p,q} = H_p(C_* \otimes H_q(C'_*)) = H_p(C_*) \otimes H_q(C'_*).$$

Now an element of $E^2_{p,q}$ can be represented by a sum of elements of the form $\alpha \otimes \beta$ where α is a cycle in C_p and β is a cycle in C'_q. This is a cycle in $C \otimes C'$, hence by definition, all higher differentials in the spectral sequence vanish. Thus $E^\infty_{p,q} = E^2_{p,q}$. It follows readily that the obvious map

$$\bigoplus_{i+j=k} H_i(C_*) \otimes H_j(C'_*) \longrightarrow H_{i+j}(C \otimes C')$$

is an isomorphism.

4 The Leray-Serre spectral sequence for homology

We now introduce a spectral sequence which relates the homology of a bundle (or more generally, a Serre fibration) to the homology of the fibers and the base. In favorable cases, it allows one to compute one of these homologies if one knows the other two.

Let $\pi : E \to B$ be a Serre fibration. (See handout on homotopy theory.) Recall that the homology of the fibers forms a local coefficient system on B, which we denote by $\{H_*(E_x)\}$. (See handout on homotopy theory.) Recall also that if B is simply connected, then every local coefficient system on B is constant.

Theorem 4.1 *Let $\pi : E \to B$ be a Serre fibration. Then there exists a spectral sequence $E^r_{p,q}$, defined for $r \geq 2$, with*

$$E^2_{p,q} = H_p(B; \{H_q(E_x)\}),$$

and converging to

$$E^\infty_{p,q} = G_p H_{p+q}(E)$$

for some filtration on $H_*(E)$.

We postpone the construction of the spectral sequence, and first consider some examples and applications.

Example 4.2 [compute homology of $SU(4)$]

Example 4.3 [discuss homology of S^1-bundles over S^2, and relate ∂_2 to the Euler number]

Example 4.4 [prove the Hurewicz isomorphism using the path fibration]

Proof of Theorem 4.1. We now sketch two constructions of the Leray-Serre spectral sequence.

The first construction only works in the special case when B is a CW complex. Let B^p denote the p skeleton of B, and let $C_*(E)$ denote the singular chain complex of E. Define a filtration on $C_*(E)$ by setting $F_pC_*(E)$ to be the subcomplex consisting of singular chains supported in $\pi^{-1}(B^p)$. This then gives rise to a spectral sequence. By definition, the associated graded chain complex is $G_pC_*(E) = C_*(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$, so the E^1 term of the spectral sequence is the relative homology

$$E^1_{p,q} = H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})).$$

We can calculate this relative homology as a direct sum over the p-cells $\sigma : D^p \to B$ of $H_{p+q}(\sigma^* E, (\sigma|_{S^{p-1}})^* E)$ to find that

$$E^1_{p,q} = C^{cell}_p(B; \{H_q(E_x)\}).$$

With some work, generalizing Example 3.3, one can show that the differential ∂_1 on the left hand side corresponds to the cellular differential with local coefficients on the right hand side, so that

$$E^2_{p,q} = H_p(B; \{H_q(E_x)\}).$$

As in Example 3.3, the spectral sequence converges to $G_p H_{p+q}(E)$.

To give a second construction of the Leray-Serre spectral sequence when B is not necessarily a CW complex, let $C_*(E)$ denote the cubical singular chain complex of E. (See handout on homotopy theory.) Define a filtration on $C_*(E)$ by setting $F_p C_{p+q}(E)$ to be the span of those singular cubes $\sigma : I^{p+q} \to E$ such that the projection $\pi \circ \sigma : I^{p+q} \to B$ is independent of the last q coordinates on I^{p+q}. Note that such a cube descends to a "horizontal" p-cube $\sigma_h : I^p \to B$. Also, restricting to the center of I^p defines a "vertical" q-cube $\sigma_v : I^q \to E_x$, where $x \in B$ is the center of σ_h. The horizontal and vertical cubes define a map

$$F_p C_{p+q}(E) \longrightarrow \bigoplus_{\sigma_h : I^p \to B} C_q(E_{\mathrm{center}(\sigma_h)}).$$

If we mod out by degenerate[1] cubes σ_h on the right hand side, then we obtain a well-defined map

$$\Phi_0 : E^0_{p,q} = G_p C_{p+q}(E) \longrightarrow \bigoplus_{\{\sigma_h : I^p \to B \text{ nondegenerate}\}} C_q(E_{\mathrm{center}(\sigma_h)}).$$

Now the differential ∂_0 only considers the "vertical" boundary. That is, $\partial_0(\sigma)$ is a signed sum of those faces of σ in which one omits one of the last q coordinates. Thus if $\Phi_0(\sigma) = (\sigma_h, \sigma_v)$, then

$$\Phi_0(\partial_0 \sigma) = (-1)^q (\sigma_h, \partial \sigma_v).$$

Therefore Φ_0 induces a map on homology

$$\Phi_1 : E^1_{p,q} \longrightarrow \bigoplus_{\{\sigma_h : I^p \to B \text{ nondegenerate}\}} H_q(E_{\mathrm{center}(\sigma_h)}) = C_p(B; \{H_q(E_x)\}).$$

[1] For the purposes of this discussion, we will declare a cube to be degenerate if it is independent of the *last* coordinate.

Now one can use the homotopy lifting property for cubes to define a right inverse to Φ_1. That is, given a p-cube σ_h in B and a q-dimensional cycle in the fiber over its center, one needs to realize this as a linear combination of $(p+q)$-cubes covering σ_h. We omit the details. One can further use the homotopy lifting property to show that the right inverse to Φ_1 so constructed is also a left inverse. Thus Φ_1 is an isomorphism.

Now ∂_1 is given by the "horizontal" component of the differential, in which one omits one of the first p coordinates of the cube. It then follows from the definition of the "parallel transport" in the local coefficient system $\{H_q(E_x)\}$ that ∂_1 agrees with the differential on $C_p(B; \{H_q(E_x)\})$. Therefore

$$E^2_{p,q} = H_p(B; \{H_q(E_x)\}).$$

Since the filtration of $C_i(E)$ is bounded between -1 and i, the spectral sequence converges to

$$E^\infty_{p,q} = G_p H_{p+q}(E).$$

(When B is a CW complex, the second construction gives the same spectral sequence as the first from the E^2 term on. We omit the proof.) □

Example 4.5 [recover Eilenberg-Zilber theorem]

Example 4.6 [Show from the definition that for a circle bundle over S^2, ∂_2 agrees with the Euler number.]

Example 4.7 [Explain why an element of $E^2_{p,0}$ survives to E^∞ iff it is in the image of π_*.]

5 Cohomological spectral sequences and products

A cohomological spectral sequence is defined as above but with the arrows reversed. Namely we have R-modules $E_r^{p,q}$ defined for $r \geq r_0$ and differentials

$$\delta_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1},$$

such that E_{r+1} is the homology of (E_r, δ_r). A cochain complex $(C^*, \delta : C^* \to C^{*+1})$, together with a *decreasing* filtration $F_pC_* \supset F_{p+1}C_*$, gives rise to a spectral sequence

$$E_r^{p,q} = \frac{\{x \in F_pC^{p+q} \mid \partial x \in F_{p+r}C_{p+q+1}\}}{F_{p+1}C_{p+q} + \partial(F_{p-r+1}C_{p+q-1})}.$$

This spectral sequence has $E_1^{p,q} = H^{p+q}(G_p C^*)$, and converges to $G_p H^{p+q}(C_*)$ if the filtration of each C^i is bounded.

One advantage of cohomology over homology is that we can consider products. Suppose our filtered cochain complex is equipped with a product $\star : C^i \otimes C^j \to C^{i+j}$ such that:

- δ is a derivation with respect to the product, i.e. for $\alpha \in C^i$ and $\beta \in C^k$ we have
$$\delta(\alpha \star \beta) = (\delta\alpha) \star \beta + (-1)^i \alpha \star (\delta\beta).$$

- The product respects the filtration, in that
$$\star : F_p C^* \otimes F_{p'} C^* \longrightarrow F_{p+p'} C^*.$$

The above assumption implies that $\star$ induces a well-defined product
$$\star_0 : \frac{F_p C_*}{F_{p+1} C_*} \otimes \frac{F_{p'} C_*}{F_{p'+1} C_*} \longrightarrow \frac{F_{p+p'} C_*}{F_{p+p'+1} C_*}.$$

More generally, it is easy to see that $\star$ induces a well-defined map
$$\star_r : E_r^{p,q} \otimes E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$
sending $[x] \otimes [y] \longmapsto [x \star y]$.

Proposition 5.1 *Under the above assumptions, the products $\star_r$ have the following properties:*

- *δ_r is a derivation with respect to $\star_r$:*
$$\delta_r(\alpha \star_r \beta) = (\delta_r \alpha) \star_r \beta + (-1)^{p+q} \alpha \star_r (\delta_r \beta).$$

- *$\star_{r+1}$ is the product on the homology of (E_r, δ_r) induced by $\star_r$.*

- *If the filtration of each C^i is bounded, then the limiting product*
$$\star_\infty : G_p H^i \otimes G_{p'} H^j \longrightarrow G_{p+p'} H^{i+j}$$
is the top graded piece of the product
$$\star : F_p H^i \otimes F_{p'} H^j \longrightarrow F_{p+p'} H^{i+j}.$$

Proof. Exercise. □

6 Cohomological Leray-Serre and cup product

If $(F_p C_*, \partial)$ is a chain complex with a (bounded) increasing filtration, then the dual chain complex $\operatorname{Hom}(C_*, R)$ has a (bounded) decreasing filtration defined by setting $F_p \operatorname{Hom}(C_*, R)$ to be the annihilator of $F_{p-1}C_*$. If $F_{p-1}C_*$ is a free summand of $F_p C_*$, then we have

$$G_p \operatorname{Hom}(C_*, R) = \frac{\operatorname{Ann}(F_{p-1}C_*)}{\operatorname{Ann}(F_p C_*)} = \operatorname{Hom}(G_p C_*, R).$$

Thus we obtain a cohomological spectral sequence with

$$E_1^{p,q} = H_{p+q}(\operatorname{Hom}(G_p C_*, R)),$$

and the differential on E_1 is obtained by applying $\operatorname{Hom}(-, R)$ to the differential on the E^1 page of the homological spectral sequence for C_*.

If $\pi : E \to B$ is a Serre fibration, then applying the above discussion to either construction of the homological Leray-Serre spectral sequence gives a cohomological version of the Leray-Serre spectral sequence, with

$$E_2^{p,q} = H^p(B; \{H^q(E_x; R)\})$$

which converges to

$$E_\infty^{p,q} = G_p H^{p+q}(E; R).$$

We now consider products. It follows immediately from the definition that the cup product[2] on $C^*(E; R)$ respects the second filtration on $C^*(E; R)$. (One can also check this with a bit more work for the first filtration on $C^*(E; R)$ when B is a CW complex.) Hence we have products $\star_r$ on E_r for which δ_r is a derivation. We now describe the product

$$\star_2 : H^p(B; \{H^q(E_x; R)\}) \otimes H^{p'}(B; \{H^{q'}(E_x; R)\}) \longrightarrow H^{p+p'}(B; \{H^{q+q'}(E_x; R)\}).$$

Recall that if $\mathscr{G}$ and $\mathscr{G}'$ are two local coefficient systems on B, then the definition of cup product generalizes in a straightforward manner to give a cup product with local coefficients

$$\smile : H^p(B; \mathscr{G}) \otimes H^{p'}(B; \mathscr{G}') \longrightarrow H^{p+p'}(B; \mathscr{G} \otimes \mathscr{G}').$$

[2]Recall that the cup product of cubical cochains $\alpha \in C^i$ and $\beta \in C^j$ is defined as follows. Given an $(i+j)$-cube σ, we define $(\alpha \cup \beta)(\sigma)$ to be an appropriately signed sum of all products $\alpha(\sigma_J^0)\beta(\sigma_I^1)$. Here J and I are complementary subsets of $\{1, \ldots, i+j\}$ of cardinality j and i respectively; the cube σ_J^0 is obtained by setting all coordinates in J equal to 0; and the cube σ_I^1 is obtained by setting all coordinates in I equal to 1.

Taking $\mathscr{G} = \{H^q(E_x; R)\}$ and $\mathscr{G}' = \{H^{q'}(E_x; R)\}$, and composing with the cup product on the fibers, we obtain a cup product

$$\smile: H^p(B; \{H^q(E_x; R)\}) \otimes H^{p'}(B; \{H^{q'}(E_x; R)\}) \longrightarrow H^{p+p'}(B; \{H^{q+q'}(E_x; R)\}).$$

We claim now that if $\alpha \in E_2^{p,q}$ and $\alpha' \in E_2^{p',q'}$, then

$$\alpha \star_2 \alpha' = (-1)^{qp'} \alpha \smile \alpha' \in E_2^{p+p',q+q'}.$$

[Need proof.]

Example 6.1 In some cases, the product structure drastically simplifies computations using the Leray-Serre spectral sequence.

[compute cohomology ring of $SU(n)$]

Example 6.2 [Thom isomorphism]

Example 6.3 [sphere bundles]

Example 6.4 [Leray-Hirsch theorem]

7 The universal coefficient spectral sequence

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